Energetic Consistency and Momentum Conservation in the Gyrokinetic Description of Tokamak Plasmas

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Abstract

Gyrokinetic field theory is addressed in the context of a general Hamiltonian. The background magnetic geometry is static and axisymmetric, and all dependence of the Lagrangian upon dynamical variables is in the Hamiltonian or in free field terms. Equations for the fields are given by functional derivatives. The symmetry through the Hamiltonian with time and toroidal angle invariance of the geometry lead to energy and toroidal momentum conservation. In various levels of ordering against fluctuation amplitude, energetic consistency is exact. The role of this in underpinning of conservation laws is emphasised. Local transport equations for the vorticity, toroidal momentum, and energy are derived. In particular, the momentum equation is shown for any form of Hamiltonian to be well behaved and to relax to its magnetohydrodynamic (MHD) form when long wavelength approximations are taken in the Hamiltonian. Several currently used forms, those which form the basis of most global simulations, are shown to be well defined within the gyrokinetic field theory and energetic consistency.

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I. INTRODUCTION

Gyrokinetic theory is a well founded formalism by which particle motion is treated in terms of drifts of particle gyrocenters rather than the combination of gyromotion and drifts of the particles. Particle motion in a magnetic field is set up with a drift kinetic Lagrangian assuming arbitrarily large magnetic field scale length [1–3], and then Lie transform techniques assuming small product of gyroradius and field amplitude are applied to obtain a Lagrangian still independent of gyrophase angle but valid for a gyroradius, while still small compared to background scale lengths, of arbitrary order with respect to the scale of E-cross-B eddies [4–7]. Alternatively, the field variable amplitude may be left arbitrary when the small parameter for expansion is the local ratio between the gyroradius and the scale of potential variation [8, 9]. Initially in this formulation, a back transformation was used to obtain the self consistent field polarisation equation for the electrostatic potential [5, 6]. Taken together the gyrokinetic Lie transform and this back transform are an application of push forward and pull back transforms in differential geometry [4, 10, 11]. More recently the entire Lagrangian is set up as the integral of a Lagrangian density over the phase space, with the polarisation equation obtained as the Euler-Lagrange equation by varying the electrostatic potential in the field Lagrangian [12, 13]. These two approaches were shown to be equivalent in the recent review by Brizard and Hahm [11]. Moreover, in an analysis of gyrokinetic transformation of the general Landau collision operator, the method of Lie transforming the Lagrangian and deriving the Euler-Lagrange equations was shown to be equivalent to the Poisson bracket transform of the Vlasov or Boltzmann kinetic equation directly, with the latter method able to treat collisional dissipation but with the Lie transform useful in deciding which coordinate map to use [14]. Following demonstrative gyrokinetic simulation of the internal kink mode [15], gyrokinetic theory was explicitly linked back to MHD for long wavelength electromagnetic oscillations and instabilities [16–19], Using the large amplitude/long wavelength form of Ref. [9] the correspondence to nonlinear reduced magnetohydrodynamics (MHD) and with the long wavelength limit of the small amplitude version was shown at the level of the Lagrangians. The theory is now a fully self consistent Lagrangian field theory. As it does not depend on assumptions concerning the form of the distribution function, it is also necessarily a total-f formulation.

Energy conservation has been well known since the development of the Lie transform

version of the theory as cited above. Momentum has received less attention despite the general demonstration of conservation via the Noether theorem [12]. However, since the rise of gyrokinetic treatments of neoclassical flows [20, 21], discussion of the Coriolis drift and turbulent equipartition effects [22–25] has emerged during observation of the tokamak momentum pinch [26]. The turbulent equipartition scenario as an indirect effect involving exchange channels more than actual drive effects especially underscores the role of strict conservation in a complicated, nonlinear physical situation. Momentum conservation was also demonstrated for evolution of axisymmetric flows and currents toward equilibrium in the context of total-f electromagnetic theory and computation under edge conditions [27].

Gyrokinetic field theory was not necessary to build the original gyrokinetic computational models whose self consistent equation for the field potential was referred to as the gyrokinetic Poisson equation [28–30]. Despite recovery of these equations by the Lie transform and field theory methods cited above, the usefulness of the gyrokinetic Poisson equation for determining the electric field has been criticised [31]. Within the field theory, however, the Euler-Lagrange equation for the electrostatic potential is indeed one and the same with this gyrokinetic Poisson equation [12]. Obtaining the equation via different methods outside of the field theory would appear to break the inherent consistency unless the method were found to be equivalent to use of the field theory, as the earlier versions in fact are.

We therefore address the general question of momentum conservation within the energetic consistency afforded by the field theory version of gyrokinetics. The results of any particular Lie transform are assumed to be given, with the only stipulation being the general form of a Lagrangian in which all dependence on time dependent field variables is transformed into the Hamiltonian (*i.e.*, the "interaction Lagrangian" as described in basic texts [32]). Both energy and momentum are considered, with the role of symmetries as paramount. In a tokamak under low frequency conditions, the conserved momentum is toroidal momentum; the other components of the momentum vector use the spatially dependent background magnetic field as an anchor. We leave the issue of a conserved momentum vector, a tensor transport flux, and the anchor to future work and concentrate on toroidal momentum. The result is that momentum transport and conservation yields familiar content and transport fluxes, generally and even in the specific cases of conventional models. The route back to MHD is explicitly shown via appropriate choice of the Lagrangian/Hamiltonian. The main results are valid for any ordering scheme which might be used since their demonstration

does not depend on ordering but uses general functional form of the Hamiltonian on the field variables and a field term describing shear-Alfvén disturbance magnetic energy.

Following sections of the paper describe the field theory model in general with emphasis on energetic consistency, recover global conservation laws via the Noether theorem, address the conservation of generalised vorticity, and then address toroidal momentum and energy. A 4-dimensional antisymmetric bracket form of the gyrokinetic equation is derived (the 5th and 6th dimensions do not enter due to the conservation of generalised magnetic moment and the lack of dependence by any dependent variable on gyrophase angle), which not only greatly facilitates the mathematics but is also suitable for computations [27]. Two comments on orderings are given, both how naïve ordering schemes can violate energetic consistency, and on a result showing that momentum evolution via small fluctuations occur in two places via terms of the same order (*i.e.*, orderings that occur there enter order by order in the same term multiplied by factors of order unity). The general momentum and energy transport equations that are derived are independent of any ordering. The route back to MHD is shown using what can be called an "MHD Hamiltonian" is shown explicitly for momentum and a mean field fluctuation model within that is given. The salient mathematics regards application of functional derivatives, for which the background and main operations are given in Appendix A. Then, Appendix B treats conventional models and their equivalence to gyrokinetic field theory versions in each case via appropriate choice of the Lagrangian. The results on momentum depend on a cancellation mandated by the transport equation for generalised vorticity; Appendix C gives the version of this by considering the torque due to a charge source under quasineutral conditions.

II. DESCRIPTION OF THE GYROKINETIC FIELD THEORY MODEL

The general gyrokinetic theory follows from Lie transforms applied to the extended phase space Lagrangian for the particle gyrocenter motion [4–6]. A phase space kernel locating the particles transforms the Lagrangian to a Lagrangian density [12]. In the electrostatic version the phase space integral over this density is the entire action. In the case of an electromagnetic model the magnetic energy is added as a free field contribution as in general electrodynamics (for background see the text by Landau and Lifshitz [32]). The electric field energy contribution is neglected since in a magnetised plasma the ExB kinetic energy of the

particle drifts is much larger; this is the same statement as quasineutrality since a zero space charge density is the natural result [12]. Variation of this action with respect to the gyrocenter coordinates gives the Euler-Lagrange equations for the particles. Liouville's theorem is used to convert these into an equation for a distribution function, which serves as the gyrokinetic Vlasov equation. Variation of the action with respect to the field potentials gives the polarisation and induction equations giving the self consistent response of the electrostatic potential and parallel magnetic potential, respectively [12, 13].

Gyrokinetic theory was recently given in terms of the field theory for transonic ExB flows [9], readdressing the large amplitude (strong ExB flow) version of Ref. [8]. Following Ref. [33], the ExB flow resulting from the appearance of the large-scale potential in the lowest-order Euler-Lagrange equations was used in the coordinate transformations, rather than a background flow given by a fluid analysis. In our case, however, the same potential was used for all flow dynamics without splitting the fluctuations from the background; the dependent field variable was used for both purposes. The choice of Lie transform was then changed to move all effects from the flow potential into the Hamiltonian. The model was shown to recover the conventional ones [6, 7], including nonlinear reduced MHD [34, 35], for weaker flow amplitude and larger scale.

The particle Lagrangian itself starts with the form from electrodynamics and is Lie transformed into a low-frequency form in an expansion which formally uses the amplitude of the drifts as a small parameter; this can be either the fluctuation amplitude as in Refs. [5, 6] or the gyroradius compared to the dynamical scale length as in Refs. [8, 9].

The Lie transform method is rather general, and various choices exist, but for present purposes it is useful to know that the choice can always be made to arrange the particle Lagrangian, L_p, so that the symplectic part (the vector of coefficients, p_i, of particle coordinate time derivatives, $\dot{q}^i$, in the representation $L_p = p_i\dot{q}^i - H$) depends on background geometry only, and all time dependent field effects appear in the Hamiltonian, H, only. Since the latter serves as the time component of the underlying fundamental one-form, this casts the Euler-Lagrange equations in a form where partial time derivatives on dynamical field variables are absent. Resulting geometric quantities, including volume elements and Jacobians, are strictly static. This was the Lie-transform strategy used in Ref. [9], emphasising the general desirability of arranging L_p with all time dependence in H not only for computations but also in proving correspondence to other forms such as nonlinear reduced

MHD. For the case of tokamak geometry, this has the added benefit of isolating all toroidal angle dependence into H as well, which will be seen to facilitate the proof of results involving toroidal momentum conservation.

We therefore assume a particle Lagrangian which for any degree of expansion has been Lie transformed into the following form

$$L_p = \left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b}\right)\cdot\dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - H \tag{1}$$

where $\{\mathbf{Z}_p\} = \{\mathbf{R}, p_z, \mu\}$ are the particle coordinates (gyrocenter position, parallel canonical momentum, gyration magnetic moment), ϑ is the ignorable sixth coordinate giving the gyrophase angle, e and m are the species charge and mass, $\mathbf{A}$ and $\mathbf{b}$ are the potential and unit vector for $\mathbf{B} = \nabla\times\mathbf{A}$ the background magnetic field, and H is the Hamiltonian. The canonical parallel momentum version is used, so that the time dependent parallel magnetic potential $A_\parallel$ appears only in H. All flow dynamics due to the time dependent electrostatic potential ϕ appear also only in H. Here and below, the parallel subscript denotes the component locally parallel to $\mathbf{B}$.

The Hamiltonian depends on both field potentials ϕ and $A_\parallel$, evaluated at the gyrocenter positions via, e.g., $\phi(\mathbf{R})$, as well as the gyrocenter phase space coordinates,

$$H = H(\mathbf{R}, p_z, \mu, \phi, A_\parallel) \tag{2}$$

The parallel velocity U is not used explicitly as a coordinate but can be defined as a derivative of H,

$$U \equiv \frac{\partial H}{\partial p_z} \tag{3}$$

It is important in all the derivations to note that U has spatial and time derivatives through its dependence on $A_\parallel$.

The dependence of H upon the fields $\phi, A_\parallel$ involves differential operators such as spatial derivatives which commute generally with variations or spatial or time derivatives, or the gyroaveraging operator J_0 which depends on μ and B as well as spatial derivatives. One has to know whether these operators commute with derivatives. With all time and toroidal angle dependence transformed into H we ensure to be working with a representation in which this is true generally for differentiation with respect to either time or toroidal angle although not for all components of the gradient operator.

Formally, J_0 has the form in wavenumber space of multiplication of Fourier coefficients by the zeroth Bessel function $J_0(k_\perp \rho_L)$, where ρ_L is the particle gyroradius given by $\rho_L = v_\perp / |eB/mc|$, with gyrofrequency $|eB/mc|$, or in terms of the coordinates by $\rho_L^2 = 2\mu B/[m(eB/mc)^2]$. The perp subscript denotes the component in the plane locally perpendicular to $\mathbf{B}$. Hence J_0 is time symmetric in any geometry but toroidal angle symmetric only in tokamak geometry. The J_0 operator may be cast as a series of perpendicular Laplacians, $\nabla_\perp^2$, so in the local transport equations to be derived below it is sufficient to consider H with arbitrary dependence on field amplitude and the field gradient and Laplacian. This becomes necessary when considering the role of functional derivatives in the theory.

The particle equations of motion are found from the Euler-Lagrange equations resulting from L_p. The drift motion,

$$B_\parallel^* \frac{d\mathbf{R}}{dt} = \nabla H \cdot \frac{c}{e}\frac{\mathbf{F}}{B} + U\mathbf{B}^* \qquad\qquad B_\parallel^* \frac{dp_z}{dt} = -\mathbf{B}^* \cdot \nabla H \tag{4}$$

separates naturally from the gyromotion,

$$\frac{d\mu}{dt} = 0 \qquad\qquad \frac{d\vartheta}{dt} = \frac{e}{mc}\frac{\partial H}{\partial \mu} \tag{5}$$

where some standard definitions are

$$\mathbf{A}^* = \mathbf{A} + p_z \frac{c}{e}\mathbf{b} \qquad\qquad \mathbf{B}^* = \nabla \times \mathbf{A}^* \qquad\qquad B_\parallel^* = \mathbf{b} \cdot \mathbf{B}^* \tag{6}$$

Drift tensor notation is used, with

$$\mathbf{F} = \nabla \mathbf{A} - (\nabla \mathbf{A})^T \tag{7}$$

where superscript T denotes the transpose. It follows that

$$\mathbf{F} = \epsilon \cdot \mathbf{B} \qquad\qquad \nabla \times \mathbf{b} = -\nabla \cdot \frac{\mathbf{F}}{B} \qquad\qquad \mathbf{B}^* = \mathbf{B} - p_z \nabla \cdot \frac{c}{e}\frac{\mathbf{F}}{B} \tag{8}$$

where ϵ is the rank-three Levi-Civita pseudotensor. Phase space volume conservation is expressed by

$$\frac{\partial}{\partial \mathbf{Z}_p} \cdot (\sqrt{g} B_\parallel^* \dot{\mathbf{Z}}_\mathbf{p}) = 0 \tag{9}$$

where $\sqrt{g}$ is the determinant of covariant components of the coordinate metric. With $\nabla \cdot \mathbf{B}^* = 0$, Eq. (9) implies

$$\nabla \cdot \frac{c}{e}\frac{\mathbf{F}}{B} + \frac{\partial \mathbf{B}^*}{\partial p_z} = 0 \tag{10}$$

Guaranteed by the definition of $\mathbf{B}^*$, this also determines that the quantity $B_\parallel^*$ serves as the volume element of the velocity space, where the volume element of the entire phase space is $\sqrt{g}B_\parallel^*$.

This particle Lagrangian is converted to a system one by placing the particles in phase space via the kernel $G(\mathbf{Z}_p, \mathbf{Z})$ with

$$\mathbf{Z} \to \{\mathbf{x}, z, w\} \qquad\qquad \mathbf{Z}_p \to \{\mathbf{R}, p_z, \mu\} \tag{11}$$

giving the correspondence between phase space coordinates $\mathbf{Z}$ and gyrocenter coordinates $\mathbf{Z}_p$, respectively. Due to some notational difficulties with the rest of this work, however, we will dispense with this distinction between $\mathbf{Z}$ and $\mathbf{Z}_p$, leaving the role of G to be understood once we already have the particle equations of motion in Eqs. (4,5).

The phase space integral is denoted $\int d\Lambda$. The integration domain $d\Lambda$ is given as a combination of the velocity space and configuration space domains. These are given respectively by

$$d\Lambda = d\mathcal{V} \otimes d\mathcal{W} \qquad\qquad d\mathcal{V} = \sqrt{g}\, dx^1\, dx^2\, dx^3 \qquad\qquad d\mathcal{W} = 2\pi m^{-2}\, dp_z\, d\mu\, B_\parallel^* \tag{12}$$

where $\sqrt{g}$ is the determinant of covariant components of the coordinate metric, and noting that the form of $d\mathcal{W}$ is determined by phase space conservation (the $B_\parallel^*$ factor) and normalisation (the $2\pi m^{-2}$ factor). The Lagrangian for the entire particles/field system is then

$$L = \sum_{\text{sp}} \int d\Lambda\, f\, L_p - \int d\mathcal{V}\, \frac{B_\perp^2}{8\pi} \tag{13}$$

where the sum is over species. The electrodynamic field term $(E^2 - B^2)/8\pi$ reduces to $-B_\perp^2/8\pi$ as the assumption of quasineutrality eliminates $E^2/8\pi$ in favour of the ExB kinetic energy of the particle drifts, and the assumptions of low frequency $\omega \ll k_\perp v_A$ and low plasma beta $\beta = 8\pi p/B^2 \ll 1$ restrict magnetic variation to the parallel magnetic potential $A_\parallel$. A simplified version usable for the present purposes is

$$B_\perp^2 = \left|\nabla_\perp A_\parallel\right|^2 \tag{14}$$

defined with the perpendicular spatial derivatives. Hence $B_\perp^2/8\pi$ is identified as the energy in shear Alfvén magnetic disturbances perpendicular to the equilibrium magnetic field. More general forms are possible (cf. Appendix A) but the field term is always quadratic in $A_\parallel$.

Under conditions of magnetic compressibility, $\mathbf{A}_\perp$ enters as well (cf. Sec. III C of Ref. [11]), but the structure of the theory remains as presented herein.

The gyrokinetic Vlasov equation is found via variation of the $\mathbf{Z}_p$ components according to characteristic methods [12], or equivalently by application of Liouville's theorem to the particle motion in Eqs. (4,5), yielding

$$B_\parallel^* \frac{\partial f}{\partial t} + \nabla H \cdot \frac{c}{e}\frac{\mathbf{F}}{B} \cdot \nabla f + \mathbf{B}^* \cdot \left(\frac{\partial H}{\partial p_z} \nabla f - \frac{\partial f}{\partial p_z} \nabla H \right) = 0 \tag{15}$$

Derivatives with respect to μ or ϑ do not appear, because f and H are independent of ϑ, and μ is conserved in the gyromotion. Once we have the gyrokinetic equation in this form, the distinction between phase space and gyrocenter coordinates may be left implicitly understood, since we no longer consider particles or gyrocenters as discrete entities.

The term "drifts" refers to the drift motion described by Eq. (4), especially the spatial part $\dot{\mathbf{R}}$. The part resulting from the field dependent variables is entirely contained in H. Hence when we refer to the treatment of drifts we mean the construction of H and in particular drifts to a certain order means the contributions to H due to an expansion up to that order. The results we will obtain do not depend on the form of ordering (just on the functional form of the dependence of H upon ϕ and $A_\parallel$), but at certain points we will need to refer to the result in terms of a certain ordering.

The equations for the fields are determined by functional derivatives (cf. Appendix A and the background references cited there). The self consistent polarisation equation (also called gyrokinetic Poisson equation) is given by the Euler-Lagrange equation for ϕ from this same Lagrangian. It is found by varying the Lagrangian with respect to ϕ, yielding an integral over $d\mathcal{V}$ of $\delta\phi(\mathbf{x})$ times a coefficient, which is required to vanish. It is the same statement as requiring the functional derivative of L with respect to ϕ to vanish. This produces

$$\sum_{\text{sp}} \frac{\delta f H}{\delta \phi} = 0 \tag{16}$$

The species-summed functional derivative of fH vanishes alone because ϕ appears only in H. The functional derivative implies velocity space integration because it is defined with respect to the space covered by $d\mathcal{V}$. The functional derivative combination yields the gyrokinetic charge density for the particular H used (the assumption of quasineutrality sets this to zero).

The self consistent induction equation (also called gyrokinetic Ampère equation) is obtained by variation of the field potential $A_\parallel$. It is the same statement as requiring the

functional derivative of L with respect to $A_\parallel$ to vanish. This produces

$$\sum_{\rm sp} \frac{\delta f H}{\delta A_\parallel} = \frac{1}{4\pi}\nabla_\perp^2 A_\parallel \tag{17}$$

with the field term appearing on the right side arising from the field term $-B_\perp^2/8\pi$ in L. The functional derivative combination yields the gyrokinetic current (times $-1/c$) for the particular H used.

A. Antisymmetric Bracket Form of the Gyrokinetic Equation

It has been found previously that maximal symmetry in the representation of the gyrokinetic equation (Eq. 15) is helpful to the understanding of the conservation laws [27]. We observe that

$$\frac{\partial \mathbf{A}^*}{\partial p_z} = \frac{c}{e}\mathbf{b} \qquad \text{hence} \qquad \frac{\partial}{\partial p_z}\epsilon\cdot\mathbf{A}^* = \frac{c}{e}\frac{\mathbf{F}}{B} \tag{18}$$

If we define

$$\mathbf{G} = \epsilon\cdot\mathbf{A}^* \tag{19}$$

we may recast Eq. (15) as

$$B_\parallel^*\frac{\partial f}{\partial t} + \nabla H\cdot\frac{\partial \mathbf{G}}{\partial p_z}\cdot\nabla f + (-\nabla\cdot\mathbf{G})\cdot\left(\frac{\partial H}{\partial p_z}\nabla f - \frac{\partial f}{\partial p_z}\nabla H\right) = 0 \tag{20}$$

This has the structure of one 3-bracket of indices $\{abz\}$

$$[H, G^{ab}, f]_{azb} = \frac{\partial G^{ab}}{\partial p_z}[H,f]_{ab} + (\nabla_a G^{ab})[H,f]_{bz} + (\nabla_b G^{ab})[H,f]_{za} \tag{21}$$

for each pair of spatial coordinates $\{ab\}$ with index z denoting the p_z coordinate. This can also be written as

$$[H, G^{ab}, f]_{azb} = \epsilon^{abc}\left(\frac{\partial A_c^*}{\partial p_z}[H,f]_{ab} + (\nabla_a A_c^*)[H,f]_{bz} + (\nabla_b A_c^*)[H,f]_{za}\right) \tag{22}$$

where on the right side the Einstein summation convention is used for repeated (up/down) indices. In each case the 2-bracket form is

$$[H,f]_{ab} = H_{,a}\,f_{,b} - H_{,b}\,f_{,a} \tag{23}$$

with the comma denoting differentiation with respect to the coordinate whose index is given by the subscript. Since there is no p_z-component of $\mathbf{A}^*$ we may add 3 more fictitious

3-brackets, one for each pair of spatial indices and p_z with A_z^*. The entire combination becomes

$$\frac{\partial f}{\partial t} + \mathcal{E}^{abcd} H_{,a} f_{,b} A^*_{c,d} = 0 \tag{24}$$

where $\mathcal{E}$ is the rank-four Levi-Civita pseudotensor in the 4-space covered by $d\mathcal{V} \otimes dp_z$. The components of ϵ^{abc} are $1/\sqrt{g}$ times ± 1 or 0 depending on the permutation of spatial indices $\{abc\}$. The components of $\mathcal{E}^{abcd}$ are $1/\sqrt{g} B^*_\parallel$ times ± 1 or 0 depending on the permutation of indices $\{abcd\}$ in the 4-space domain. The 3-space order is $\{123\}$ for $dx^1\, dx^2\, dx^3$ and hence the 4-space order is $\{123z\}$ for $dx^1\, dx^2\, dx^3\, dp_z$. Positive, negative, and zero permutations of these give the other components.

It was previously observed that axisymmetric momentum conservation follows directly from this form of the gyrokinetic equation, simply due to symmetries in the indices [27]. In this work this antisymmetric bracket form will be used to facilitate proof of the conservation laws for energy and toroidal momentum for general dependence of the Hamiltonian upon the dynamical fields.

III. GLOBAL CONSERVATION LAWS

The conserved energy is found from the total action $\int dt\, L(\mathbf{Z}_p, \phi, A_\parallel, t)$ via Noether's theorem, applying small variations to the time component [32]. In the gyrokinetic case this has been done before, both from the discrete particle characteristics point of view [12], and from the continuum/field representation using constrained variations [13]. Since L is first order in all the time derivatives this becomes the combination of all the $p_i \dot{q}^i$ terms less the Lagrangian. This defines the Noether energy as

$$\mathcal{E} = \sum_{\text{sp}} \int d\Lambda\, f\, H + \int d\mathcal{V}\, \frac{B_\perp^2}{8\pi} \tag{25}$$

In the electrostatic case it is simply the integral over fH summed over species. Since the background magnetic field (through $\mathbf{A}$) is not varied, it does not appear in the Noether energy.

The same follows for the Noether momentum, by applying small variations to the space component [32]. In the gyrokinetic case this was given in abstract form by Refs. [12, 13], but not for specific cases. Working out the space components is complicated by the fact that in low frequency dynamics in a magnetised plasma, the magnetic field serves as an anchor

for momentum, so that the general four-vector version is not conserved for the dynamics under consideration: not only is the Poynting momentum neglected against the plasma momentum in ExB motion (through the assumption of quasineutrality), but also the neglect of compressional Alfvén dynamics removes the exchange with the background field. In a tokamak, only toroidal momentum is conserved. For L of the form given in Eq. (13), the conserved toroidal momentum is simply given by the toroidal canonical momentum weighted by f and summed over species. The Noether toroidal momentum is

$$\mathcal{P} = \sum_{\text{sp}} \int d\Lambda \, f \, P_\varphi \tag{26}$$

where $P_\varphi = \partial L / \partial \dot{\varphi}$ and φ is the geometric toroidal angle. This result is a consequence of all the dependence of L upon φ is in the time component (H) or in the field terms and in the latter there is no time derivative dependence.

One other consideration is that one would like a local form of the conservation law in terms of a vector momentum density, a symmetric stress tensor for momentum transport, and a vector describing the magnetic field anchor explicitly, but this has yet to be worked out. Herein, we consider energy and toroidal momentum only, and explain their conservation using the antisymmetric bracket form of the gyrokinetic equation and the functional derivatives which describe the self consistent field equations. One motivation for this is that it is possible to directly code the antisymmetric bracket form in numerical simulations, so it is then known that the form of the equations as actually used is indeed energetically consistent.

Once the Noether energy and toroidal momentum are known, appropriate operations on the equations of motion (here, the gyrokinetic equation and the self consistent field equations) may be used to construct a local form with the time derivative of an evolving energy/momentum density and the divergence of an overall transport flux [13]. We will do this herein as well, as part of the overall motivation to establish the correspondence to fluid and MHD forms.

A. Energetic Consistency

Both of these approaches lead naturally to the known results on energetic consistency, namely that the same H must be used to obtain the gyrokinetic equation and the (related)

field equations. Approximations are done in L (hence H) and then the equations are derived without further approximation. Specifically, orderings in the derivation of L are used (via Lie transforms or some other method), but not thereafter in the derivation of the Euler-Lagrange equations. Ref. [12] points out specifically that if the drift motion is to be followed with lowest-order forms of H, all higher-order forms must be cast into field terms. For example, if $H = H_0 + H_1 + H_2$, with the last piece containing the ExB energy, and it is desired to advance the gyrokinetic equation only with $H_0 + H_1$, then the term fH_2 in L must be replaced by $f_0 H_2$ where f_0 is a background static form which can be thought of as part of the geometry. Then, since H_2 does not multiply f, it is not involved in the gyrokinetic equation itself but only as a field term which would appear on the right hand side of the polarisation equation, e.g.,

$$\sum_{\rm sp} \frac{\delta}{\delta\phi} f(H_0 + H_1) = -\sum_{\rm sp} \frac{\delta}{\delta\phi} f_0 H_2 \tag{27}$$

This is referred to as *linearised polarisation*. The two assumptions go together: first order drift motion, and the appearance of f_0 in the polarisation term (the right side of Eq. 27) Conversely, if one desires to keep the dependent variable f in this term, restoring Eq. (16), then the corresponding H_2 must be kept in the drift motion. This is the basic statement of energetic consistency in a total-f global model and the essential references [12, 13] arrived at this result ten years ago. The same result is found for the same reasons in gyrofluid field theory models which have a different starting point but are also Lagrangian/Hamiltonian models [36, 37]. It is related to the connection in fluid models between advection and divergence forms of the equation of motion with respect to the polarisation drift velocity in a fluid model (why the polarisation drift must be kept in advection if the species mass density involves the dependent variable for species particle density [38]).

A clear extension of this is that in any discussion of drift motion past first order, say to order n, the f must be kept as the dependent variable in all of the terms $H_0 + H_1 + \cdots + H_n$ in the functional derivatives in the polarisation equation. Any energetic contributions $H_{n+1} + \cdots$ must then either be dropped or combined with a background f_0 in L, with the polarisation equation then becoming the appropriately generalised version of Eq. (27). Of course, if L is derived or constructed first and then the Euler-Lagrange equations are derived without approximation thereafter, then energetic consistency becomes a guaranteed result.

B. Time Symmetry and Energy Conservation

Using the antisymmetric bracket form of the gyrokinetic equation (Eq. 24) we multiply by H and use the linearity of the derivatives $Hf_{,b} = (fH)_{,b} - fH_{,b}$ and the antisymmetry (the form with $H_{,a}H_{,b}$ vanishes due to the permutation of indices in $\mathcal{E}^{abcd}$), to find

$$\frac{\partial}{\partial t}(fH) + \mathcal{E}^{abcd} H_{,a} (fH)_{,b} A^*_{c,d} = f \frac{\partial H}{\partial t} \tag{28}$$

which is the local energy equation in phase space.

Integration over phase space and summation over species yields

$$\sum_{\text{sp}} \int d\Lambda \, \frac{\partial}{\partial t}(fH) = \sum_{\text{sp}} \int d\Lambda \, f \frac{\partial H}{\partial t} \tag{29}$$

with the bracket vanishing under the integral. Under the integral the right side is replaced by functional derivatives (cf. Appendix A)

$$\sum_{\text{sp}} \int d\Lambda \, \frac{\partial}{\partial t}(fH) = \int d\mathcal{V} \sum_{\text{sp}} \frac{\delta fH}{\delta \phi} \frac{\partial \phi}{\partial t} + \int d\mathcal{V} \sum_{\text{sp}} \frac{\delta fH}{\delta A_\parallel} \frac{\partial A_\parallel}{\partial t} \tag{30}$$

The first term on the right side vanishes, due to polarisation (Eq. 16). The second is replaced by the field term in $A_\parallel$, due to induction (Eq. 17), so that

$$\sum_{\text{sp}} \int d\Lambda \, \frac{\partial}{\partial t}(fH) = \int d\mathcal{V} \frac{1}{4\pi} \nabla_\perp^2 A_\parallel \frac{\partial A_\parallel}{\partial t} \tag{31}$$

Integration of the divergence operator in $\nabla_\perp^2$ by parts then yields

$$\sum_{\text{sp}} \int d\Lambda \, \frac{\partial}{\partial t}(fH) = - \int d\mathcal{V} \frac{1}{4\pi} \nabla_\perp A_\parallel \cdot \frac{\partial}{\partial t} \nabla_\perp A_\parallel = - \int d\mathcal{V} \frac{1}{8\pi} \frac{\partial}{\partial t} \left| \nabla_\perp A_\parallel \right|^2 \tag{32}$$

Identification with $B_\perp^2$ in Eq. (14) then recovers

$$\frac{\partial}{\partial t} \left(\sum_{\text{sp}} \int d\Lambda \, f \, H + \int d\mathcal{V} \, \frac{B_\perp^2}{8\pi} \right) = 0 \tag{33}$$

which is the same as obtained using the Noether theorem (cf. Eq. 25). This is the statement of energy conservation and it is valid under time symmetry for any dependence of H upon ϕ and $A_\parallel$ given the form of L stated above.

The dependence of energy conservation upon time symmetry is contained in the step from the time derivative to the functional derivative, as $\partial/\partial t$ must commute with any of the differential operators involved in the dependence of H upon ϕ and $A_\parallel$. The requirement of energetic consistency is evident in the fact that the same H is used in the gyrokinetic equation as that whose functional derivatives appear in the polarisation and induction equations. Also, the same f must appear with each of the terms in H in all cases, or else the symmetry is broken.

C. Axisymmetry and Toroidal Momentum Conservation

In this case the properties are different. For energy, H depends on both time and toroidal angle, but the symmetry of the bracket allowed combination of (fH) there. Canonical momentum at the particle level is given by

$$P_\varphi = \frac{e}{c}A^*_\varphi = \frac{e}{c}A_\varphi + p_z b_\varphi \tag{34}$$

This form is both static and axisymmetric, but does not appear in the bracket. We multiply by P_φ to find

$$\frac{\partial}{\partial t}(fP_\varphi) + \mathcal{E}^{abcd}H_{,a}(fP_\varphi)_{,b}A^*_{c,d} = f\frac{e}{c}\mathcal{E}^{abcd}H_{,a}A^*_{\varphi,b}A^*_{c,d} \tag{35}$$

Because the two appearances of A^* in the right side appear with different indices, we may make some symmetry arguments. First, the indices b and d cannot be φ, due to axisymmetry. Second, index c cannot be φ, or else A^*_φ appears twice and the remaining permutation over indices $\{abd\}$ vanishes. Hence, index a must be φ and the others are among the other three coordinates, so that

$$\frac{\partial}{\partial t}(fP_\varphi) + \mathcal{E}^{abcd}H_{,a}(fP_\varphi)_{,b}A^*_{c,d} = f\frac{\partial H}{\partial \varphi}\frac{e}{c}\mathcal{E}^{\varphi abc}A^*_{\varphi,a}A^*_{b,c} \tag{36}$$

Of the remaining terms, index b cannot be z because there is no A^*_z, so in each term one of a or c is z while the others are the two coordinates covering the perpendicular plane, which we can label 1 and 2 (hence dx^3 in $d\mathcal{V}$ is $d\varphi$). We also observe that

$$\frac{\partial \mathbf{A}^*}{\partial p_z} = \frac{c}{e}\mathbf{b} \tag{37}$$

which cancels the (e/c) factor. Noting that the units of $\mathcal{E}^{abcd}$ are $1/\sqrt{g}B^*_\parallel$, we set the positive permutation as $\{\varphi\, z\, 1\, 2\}$ and permute the $\{z\, 1\, 2\}$ indices to find

$$\frac{e}{c}\mathcal{E}^{\varphi abc}A^*_{\varphi,a}A^*_{b,c} = \frac{1}{\sqrt{g}B^*_\parallel}\left[b_\varphi(A^*_{1,2} - A^*_{2,1}) + b_1(A^*_{2,\varphi} - A^*_{\varphi,2}) + b_2(A^*_{\varphi,1} - A^*_{1,\varphi})\right] \tag{38}$$

where we have eliminated the two (zero) terms $A^*_{\varphi,1}A^*_{z,2}$ and $A^*_{\varphi,2}A^*_{z,1}$, and replaced them with the two (zero) terms $b_2A^*_{1,\varphi}$ and $b_1A^*_{2,\varphi}$, respectively. We observe that

$$\frac{1}{\sqrt{g}}\left[b_\varphi(A^*_{2,1} - A^*_{1,2}) + b_1(A^*_{\varphi,2} - A^*_{2,\varphi}) + b_2(A^*_{1,\varphi} - A^*_{\varphi,1})\right] \equiv \mathbf{b}\cdot\nabla\times\mathbf{A}^* = B^*_\parallel \tag{39}$$

and note the switch in the order of coefficients to the expression just above. Therefore we have

$$\frac{e}{c}\mathcal{E}^{\varphi abc}A^*_{\varphi,a}A^*_{b,c} = -1 \tag{40}$$

Putting this into the right side of Eq. (36), we find

$$\frac{\partial}{\partial t}(fP_\varphi) + \mathcal{E}^{abcd} H_{,a} (fP_\varphi)_{,b} A^*_{c,d} = -f\frac{\partial H}{\partial \varphi} \tag{41}$$

which is the local toroidal momentum equation in phase space.

This is the same form as the result obtained from a canonical representation of the particle Lagrangian

$$L_p = P_\psi \dot{\psi} + P_\theta \dot{\theta} + P_\varphi \dot{\varphi} + \frac{mc}{e}\mu\dot{\vartheta} - H \tag{42}$$

written directly in terms of the coordinates $\{\psi\,\theta\,\varphi\}$ as was once usual [3]). The corresponding Euler-Lagrange equation for the toroidal angle is

$$\dot{P}_\varphi = -\frac{\partial H}{\partial \varphi} \tag{43}$$

due to the axisymmetry of the rest of L_p. Using the advection forms

$$\dot{f} = \frac{\partial f}{\partial t} + \dot{\mathbf{Z}}_\mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{Z}_p} = 0 \qquad\qquad \dot{P}_\varphi = \frac{\partial P_\varphi}{\partial t} + \dot{\mathbf{Z}}_\mathbf{p} \cdot \frac{\partial P_\varphi}{\partial \mathbf{Z}_p} = -\frac{\partial H}{\partial \varphi} \tag{44}$$

we find

$$\frac{\partial}{\partial t}(fP_\varphi) + \dot{\mathbf{Z}}_\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{Z}_p}(fP_\varphi) = -f\frac{\partial H}{\partial \varphi} \tag{45}$$

which is the same form as in Eq. (41) with the bracket recast in terms of an advection.

Returning to Eq. (41), integration over phase space and summation over species yields

$$\sum_{\text{sp}} \int d\Lambda \, \frac{\partial}{\partial t}(fP_\varphi) = -\sum_{\text{sp}} \int d\Lambda \, f\frac{\partial H}{\partial \varphi} \tag{46}$$

with the bracket vanishing under the integral. Under the integral the right side is replaced by functional derivatives (cf. Appendix A)

$$\sum_{\text{sp}} \int d\Lambda \, \frac{\partial}{\partial t}(fP_\varphi) = \int d\mathcal{V} \sum_{\text{sp}} \frac{\delta f H}{\delta \phi}\frac{\partial \phi}{\partial \varphi} + \int d\mathcal{V} \sum_{\text{sp}} \frac{\delta f H}{\delta A_\parallel}\frac{\partial A_\parallel}{\partial \varphi} \tag{47}$$

The manipulations follow the energy derivation, with $\partial/\partial t$ replaced by $\partial/\partial\varphi$. In this case the derivative of $B^2_\perp$ with respect to φ vanishes under the integral, so that

$$\frac{\partial}{\partial t} \sum_{\text{sp}} \int d\Lambda \, f \, P_\varphi = 0 \tag{48}$$

which is the same as obtained using the Noether theorem (cf. Eq. 26). This is the statement of toroidal momentum conservation and it is valid under axisymmetry for any dependence of H upon ϕ and $A_\parallel$ given the form of L stated above.

The dependence of toroidal momentum conservation upon axisymmetry is contained in the step from the toroidal angle derivative to the functional derivative, as $\partial/\partial\varphi$ must commute with any of the differential operators involved in the dependence of H upon ϕ and $A_\parallel$. The requirement of energetic consistency is evident in the same way as for energy with the same loss of consistency if the symmetry between functional derivatives and the gyrokinetic equation is broken.

D. Phase Space Continuity Forms of Energy and Toroidal Momentum Conservation

Eqs. (28,41) give the antisymmetric bracket forms of the local energy and toroidal momentum equations in phase space. Using toroidal momentum as an example, We identify

$$\mathcal{E}^{abcd} H_{,a} f_{,b} A^*_{c,d} = \dot{\mathbf{Z}}_{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{Z}_p} \tag{49}$$

Then, the phase space conservation condition (Eq. 9) can be used to express Eq. (41) as

$$\frac{\partial}{\partial t}(f P_\varphi) + \frac{1}{\sqrt{g} B^*_\parallel} \frac{\partial}{\partial \mathbf{Z}_p} \cdot (\sqrt{g} B^*_\parallel f P_\varphi \dot{\mathbf{Z}}_{\mathbf{p}}) = -f \frac{\partial H}{\partial \varphi} \tag{50}$$

which is the phase space continuity equation for toroidal momentum.

Similarly, we may express Eq. (28) as

$$\frac{\partial}{\partial t}(f H) + \frac{1}{\sqrt{g} B^*_\parallel} \frac{\partial}{\partial \mathbf{Z}_p} \cdot (\sqrt{g} B^*_\parallel f H \dot{\mathbf{Z}}_{\mathbf{p}}) = f \frac{\partial H}{\partial t} \tag{51}$$

which is the phase space continuity equation for energy.

Eq. (50) makes it obvious that $f P_\varphi$ is conserved locally in axisymmetric systems ($\partial/\partial\varphi = 0$), and globally in any geometry. In Eq. (51) the term appears instead with $\partial H/\partial t$, which eventually accounts for the magnetic energy as in Eq. (33). We will use these continuity equations to produce local transport equations for both momentum and energy later.

IV. A COMMENT ON ORDERING

Before the emergence of the Lagrangian/Hamiltonian approach to drift kinetic particle motion [1–3], in which the equation for f is built explicitly using the phase space positions of the gyrocenters as dependent variables, it was customary to start with the Vlasov (or

Boltzmann) equation for the particles themselves and apply gyroaveraging through a successive ordering [39, 40]. This causes problems, however, if applied naïvely (and strictly) to the field equations. Considering an electrostatic model ($A_\parallel = 0$), we may expand H in terms of small amplitude fluctuations ($e\phi/T_e \sim \rho_* = \rho_i/L_\perp$, where ρ_i is the thermal ion gyroradius and $L_\perp$ is the profile gradient scale length [41, 42])

$$H = H_0 + H_1 + H_2 + \cdots + H_n \tag{52}$$

where at each order n the term H_n is n-th order in ϕ. As long as f is not expanded order by order, there is no problem. The same f multiplies each H_n in turn and due to the linearity property the functional derivatives add, producing Eq. (16) term by term.

However, if f is also ordered such that f_0 is the background (usually a Maxwellian) and f_1 is the fluctuation, then there is a problem. Recall that if drift motion is included to order n in H through ϕ then the polarisation equation must include fH_n in the functional derivatives to preserve energetic consistency. The f must include both f_0 and f_1. However, if the ordering is truncated at order n then the term $\delta(f_1 H_n)/\delta\phi$ is missing. Formally, it is order $n+1$. So this $(n+1)$-th order term must be kept, but in doing so we violate the ordering scheme. If the ordering scheme is applied to expand f and keep all terms up to order n, dropping all order $n+1$ terms, then this one piece will be missing. This problem is present at any order of expansion, at the last order. The only acceptable solution for orderings is to expand H in orders but not f. That is, polarisation is not to be linearised (cf. Eq. 27) if contributions above linear order in H to the drift motion are considered. Hence any discussion of orderings in which higher order drifts (even second order) are considered should be done under full energetic consistency. As noted, the field theory version of gyrokinetics is the only straightforward way to guarantee this.

V. THE EXB VORTICITY TRANSPORT EQUATION

We may form an equation for the gyrocenter charge density by multiplying the gyrokinetic equation by the charge e for each species and summing over species. For general dependence of H upon ϕ, the terms linear in ϕ are collected and all the others are combined into a total divergence. We may separate

$$H = H_0 + e\phi + H_P \tag{53}$$

where H_0 comprises all the terms not involving ϕ and then the polarisation piece H_P may be constructed from $H - H_0 - e\phi$. All gyroaveraging corrections (e.g., from $1 - J_0$) are collected into H_P. We define the generalised vorticity Ω and the polarisation vector $\mathbf{P}$ such that

$$\nabla \cdot \mathbf{P} \equiv -\Omega \equiv \sum_{\rm sp} fe \tag{54}$$

The quantity on the right side is the gyrocenter charge density. Since the derivations are being done under strict quasineutrality ($E^2/8\pi$) was neglected as discussed around Eq. 13), the quantity on the left side balances this. It is the polarisation density as developed by other methods in Ref. [28]. The generalised vorticity is defined in this manner as a quantity sensitive to small scales which (as seen below) in the MHD limit reduces to the simple ExB vorticity.

Then the polarisation equation (Eq. 16) is

$$\nabla \cdot \mathbf{P} = -\sum_{\rm sp} \frac{\delta f H_P}{\delta \phi} \tag{55}$$

It is essential for the subsequent results to be able to write the polarisation equation in this form, with polarisation density given by a divergence.

For any H with dependence on ϕ such that the separation $H = H_0 + e\phi + H_P$ yields dependence of H_P upon ϕ only through $\nabla\phi$ and $\nabla_\perp^2\phi$, the functional derivative of fH_P is

$$\frac{\delta f H_P}{\delta \phi} = \int d\mathcal{W} \left[\nabla_\perp^2 \left(f \frac{\partial H_P}{\partial \nabla_\perp^2 \phi} \right) - \nabla \cdot \left(f \frac{\partial H_P}{\partial \nabla \phi} \right) \right] \tag{56}$$

which can be written as a divergence

$$\frac{\delta f H_P}{\delta \phi} = \nabla \cdot \int d\mathcal{W} \left[\nabla_\perp \left(f \frac{\partial H_P}{\partial \nabla_\perp^2 \phi} \right) - f \frac{\partial H_P}{\partial \nabla \phi} \right] \tag{57}$$

and then the species sum of the quantity in the square brackets is identified with $\mathbf{P}$. Since within H only H_P depends on $\nabla\phi$ or $\nabla_\perp^2\phi$ we generally have

$$\mathbf{P} = \sum_{\rm sp} \int d\mathcal{W} \left[f \frac{\partial H}{\partial \nabla \phi} - \nabla_\perp \left(f \frac{\partial H}{\partial \nabla_\perp^2 \phi} \right) \right] \tag{58}$$

and the need to be able to write Eq. (55) in that form is satisfied. The only restriction on the form of H is that all terms past first order in ϕ appear only through $\nabla\phi$ or $\nabla_\perp^2\phi$. Note, however, that there is no such restriction on $A_\parallel$.

For example, for the long-wavelength electrostatic H through second order,

$$H = \frac{p_z^2}{2m} + \mu B + e\phi - \frac{mc^2}{2B^2} \left| \nabla_\perp \phi \right|^2 \tag{59}$$

the polarisation equation is

$$\sum_{\rm sp} \int d\mathcal{W} \left[ef + \frac{1}{B_\parallel^*} \nabla \cdot B_\parallel^* \frac{fmc^2}{B^2} \nabla_\perp \phi \right] = 0 \tag{60}$$

and the polarisation vector is

$$\mathbf{P} = -\sum_{\rm sp} \int d\mathcal{W} \, \frac{fmc^2}{B^2} \nabla_\perp \phi \tag{61}$$

where we note that $d\mathcal{W}/B_\parallel^*$ commutes with spatial derivatives. In this case the species-summed velocity space integral is straightforward and we have

$$\mathbf{P} = -\rho_M \frac{c^2}{B^2} \nabla_\perp \phi \qquad\qquad \Omega = \nabla \cdot \rho_M \frac{c^2}{B^2} \nabla_\perp \phi \tag{62}$$

where ρ_M is the species sum of nm with $n = \int d\mathcal{W} f$ the species density. One sees why the gyrocenter charge density plays the role of a (negative) vorticity, since $\nabla \cdot \mathbf{P}$ is proportional to $-\nabla_\perp^2 \phi$ plus corrections due to the gradients of the densities and the magnetic field strength.

The global conservation is trivial since the phase space integral of the gyrokinetic equation conserves particles for each species. The vorticity transport equation may be written as

$$\frac{\partial}{\partial t} \langle \Omega \rangle - \frac{\partial}{\partial V} \left\langle f e \dot{V} \right\rangle = 0 \tag{63}$$

Here, the angle brackets denote the flux surface average, which is the same as the volume derivative of an integral over the volume enclosed by a particular flux surface [43]. For simplicity we assume Hamada flux coordinates $\{V \theta \zeta\}$ where V is the volume enclosed by the particular flux surface, the contravariant components of the magnetic field are functions of V only, and the poloidal and toroidal angles (respectively) are unit-cycle periodic [43–45]. This leaves $\sqrt{g} = 1$. Then, $\dot{V} = \dot{\mathbf{R}} \cdot \nabla V$ is the contravariant V-component of $\dot{\mathbf{R}}$.

We have used the property that the flux surface average of a phase space divergence annihilates the velocity coordinate derivatives (with respect to p_z and μ) and commutes the integration $\int d\mathcal{W}/B_\parallel^*$ past the spatial derivatives. Then, the flux surface average annihilates the angle derivatives in a flux coordinate representation, leaving the V-component of the drift motion and the derivative $\partial/\partial V$. Note that the flux surface average of a kinetic quantity implies the species-summed velocity space integration, which is left understood.

The introduction of the polarisation vector under the time derivative leaves this equation as a pure divergence,

$$\frac{\partial}{\partial V} \left\langle \frac{\partial P^V}{\partial t} + f e \dot{V} \right\rangle = 0 \tag{64}$$

where the superscript V denotes the contravariant V-component. This describes charge conservation in the form $\nabla \cdot \mathbf{J} = 0$, as a balance between the gyrocenter drift current (the species sum of $fe\dot{\mathbf{R}}$, including the parallel piece) and the divergence of $\mathbf{P}$, so we may also identify $\partial \mathbf{P}/\partial t$ as the polarisation current. The quantity given by the flux surface average in Eq. (64), which includes both pieces, can be taken to vanish everywhere given appropriate boundary conditions (e.g., it vanishes at the magnetic axis, where $V = 0$, due to regularity of the vector component). Hence we will also have

$$\frac{\partial}{\partial V}\gamma(V)\left\langle \frac{\partial P^V}{\partial t} + fe\dot{V} \right\rangle = 0 \tag{65}$$

where γ is any flux surface quantity (also called flux function). Specifically, this expression vanishes for $\gamma = A_\varphi$ since A_φ is the quantity whose isosurfaces define flux surfaces, for a tokamak magnetic field.

VI. A FURTHER COMMENT ON ORDERING

In the conventional gyrokinetic ordering the small parameter is equivalently $k_\parallel/k_\perp$ in the wavenumber anisotropy or the fluctuation amplitude $e\phi/T_e$, which are used interchangeably [41, 42]. (The often stated ordering of $\rho_* \ll 1$ is actually *a posteriori* as it follows from the ultimate requirement that the resulting dynamics is in the range of the diamagnetic frequency ω_*, and for this to be small compared to the ion gyrofrequency requires $\rho_* \ll 1$.) Under these conditions we may simplify expressions by using the field aligned version [46, 47] of Hamada flux coordinates [43–45]. This discussion follows the version used in Ref. [47] which includes the definitions and construction algorithms for the coordinates. Starting with $\{V\theta\zeta\}$ as above, we transform the toroidal angle only, defining $\xi = \zeta - q\theta$ where $q = q(V)$ is a flux function giving the ratio B^ζ/B^θ in the contravariant components. Then, both B^V and B^ξ vanish, and the only nonvanishing component of B is B^θ. This is defined as $B^\theta = \chi' \equiv \partial\chi/\partial V$, where $\chi = \chi(V)$ is another flux definition. The tokamak magnetic field

$$\mathbf{B} = I\nabla\varphi + \nabla A_\varphi \times \nabla\varphi \tag{66}$$

may be written as

$$\mathbf{B} = \nabla\xi\times\nabla\chi = \chi'\nabla\xi\times\nabla V \tag{67}$$

which is called a Clebsch representation [46]. The sign conventions are $\nabla R\times\nabla Z\cdot\nabla\varphi > 0$ and $\chi' > 0$. It follows that $\chi' = -2\pi\,\partial\psi/\partial V$ and that for any vector (including the gradient)

the covariant components satisfy $A_\xi = A_\zeta = A_\varphi/2\pi$. With only B^θ nonvanishing, $\partial/\partial\theta$ tracks the parallel derivative and therefore we may assume $\partial/\partial\theta \ll \partial/\partial V, \partial/\partial\xi$ due to the wavenumber ordering.

We consider the flux surface average of the toroidal momentum continuity equation (Eq. 50)

$$\cdots + \frac{\partial}{\partial V}\left\langle f\frac{e}{c}A_\varphi\left(\nabla H_f\cdot\frac{c}{e}\frac{\mathbf{F}}{BB^*_\parallel}\right)^V\right\rangle = -\left\langle f\frac{\partial H_f}{\partial\varphi}\right\rangle \tag{68}$$

where the focus is on the term in the drift motion involving A_φ/c, and H_f is the part of H which involves the fluctuations in the field variables. We may apply the flute mode ordering to the drifts term and assume $B^*_\parallel = B + O(\rho_*)$ dropping the small correction, so that

$$\left(\nabla H_f\cdot\frac{c}{e}\frac{\mathbf{F}}{BB^*_\parallel}\right)^V = \frac{c}{e}\frac{1}{\chi'}\frac{\partial H_f}{\partial\xi} \tag{69}$$

where we have used the exact equality $F^{\xi V}/B^2 = 1/\chi'$ which results from the index-raising operations on $F_{\xi V} = \chi'$ and the equality $(B/\chi')^2 = g^{\xi\xi}g^{VV} - g^{\xi V}g^{V\xi}$ giving the determinant of the perpendicular contravariant metric coefficients. Recall that $\sqrt{g} = 1$ and $F_{\xi V} = \epsilon_{\xi V\theta}B^\theta$. Substituting the drifts into Eq. (68), the factors of (c/e) cancel and we find

$$\cdots + \frac{\partial}{\partial V}\frac{A_\varphi}{\chi'}\left\langle f\frac{\partial H_f}{\partial\xi}\right\rangle = -\left\langle f\frac{\partial H_f}{\partial\varphi}\right\rangle \tag{70}$$

where the flux quantities have been taken out of the flux surface average. Finally we insert the 2π normalisation in ξ so that

$$\cdots + \frac{\partial}{\partial V}\frac{2\pi A_\varphi}{\chi'}\left\langle f\frac{\partial H_f}{\partial\varphi}\right\rangle = -\left\langle f\frac{\partial H_f}{\partial\varphi}\right\rangle \tag{71}$$

We observe that the two flux surface averages are the same term. Furthermore, for a local model we may evaluate $A_\varphi = A_{\varphi,V}(V - V_0)$ near a zero at $V = V_0$ because the amplitude of the flux is arbitrary, and use $2\pi A_{\varphi,V} = -\chi'$, so that

$$\cdots + \frac{\partial}{\partial V}\left[(V_0 - V)\left\langle f\frac{\partial H_f}{\partial\varphi}\right\rangle\right] = -\left\langle f\frac{\partial H_f}{\partial\varphi}\right\rangle \tag{72}$$

where it is evident that the two terms are at the same order for any contribution by ϕ or $A_\parallel$ to H.

It is tempting to draw the cancellation obtained by differentiating the $V_0 - V$ factor. However, the remnant is of the same order. Though this manipulation does not yield a

useful final result it still serves to show that the potentially large term involving A_φ/c gives a contribution at the same order as the terms we keep on the right side of Eq. (50). Again, applying ordering to H only, the similarity in magnitude follows order by order, as the H_n contributions on each side are similar at each order. This dispenses with any notion that the large factor A_φ/c should be accompanied by higher order drift terms due to H_f than are present in the term $-\langle f H_{,\varphi}\rangle$ on the right side. We will find it necessary to keep the second order ExB energy term to obtain useful results, but since energetic consistency and the consistency in ordering we have just derived indicate, we do not need to keep terms beyond second order in ϕ and $A_\parallel$ on the left side of Eq. (50). Again, the underlying energetic consistency yields straightforward conclusions consistent with the use of a single H (to any order) everywhere within any particular version of the gyrokinetic field theory model.

The dependence of this result on energetic consistency cannot be overemphasised. In an ordering expansion, A_φ/c as a large term potentially introduces higher order terms in H. However, application of the field equations (through functional derivatives in H) up to any particular order, which obey exact energetic consistency up to the same order, was used to arrive at the result that the two expressions involving $\langle f H_{,\varphi}\rangle$ are at the same order (each contribution to H, order by order). Had the field equations been missing terms at the highest order kept in the drifts in the right side of Eq. (50), this result would not have been obtained and we would be required to discuss spurious effects. This is why discussion of orderings in H past first order must be done in the context of energetic consistency.

VII. THE TOROIDAL MOMENTUM TRANSPORT EQUATION

We now do for toroidal momentum what we did for vorticity in Sec. V. Starting with Eq. (50), we apply species summation and velocity space integration and then the flux surface average to obtain

$$\frac{\partial}{\partial t}\langle f P_\varphi\rangle + \frac{\partial}{\partial V}\left\langle f P_\varphi \dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle \tag{73}$$

where as before $\dot{V} = \dot{\mathbf{R}}\cdot\nabla V$, the velocity space integration annihilates velocity space derivatives and $d\mathcal{W}/B^*_\parallel$ commutes past the spatial divergence. We insert the definition of P_φ in Eq. (34) and collect the A_φ terms to find

$$\frac{\partial}{\partial t}\left\langle f\frac{e}{c}A_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle f\frac{e}{c}A_\varphi\dot{V}\right\rangle + \frac{\partial}{\partial t}\langle f p_z b_\varphi\rangle + \frac{\partial}{\partial V}\left\langle f p_z b_\varphi \dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle \tag{74}$$

We pull A_φ out of the flux surface average to find

$$\frac{A_\varphi}{c}\frac{\partial}{\partial t}\langle fe\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle + \frac{\partial}{\partial t}\langle fp_z b_\varphi\rangle + \frac{\partial}{\partial V}\left\langle fp_z b_\varphi \dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle \tag{75}$$

Using Eqs. (54,55), the gyrocenter charge density is replaced by the vorticity under the time derivative, to find

$$\frac{A_\varphi}{c}\frac{\partial}{\partial t}\langle \nabla\cdot\mathbf{P}\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle + \frac{\partial}{\partial t}\langle fp_z b_\varphi\rangle + \frac{\partial}{\partial V}\left\langle fp_z b_\varphi \dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle \tag{76}$$

The divergence operator on $\mathbf{P}$ is done by parts to find

$$\begin{aligned} -\frac{\partial}{\partial t}\left\langle \frac{1}{c}\mathbf{P}\cdot\nabla A_\varphi\right\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\frac{\partial}{\partial t}\left\langle P^V\right\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle \\ + \frac{\partial}{\partial t}\langle fp_z b_\varphi\rangle + \frac{\partial}{\partial V}\left\langle fp_z b_\varphi \dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle \end{aligned} \tag{77}$$

since $\nabla = \nabla V(\partial/\partial V)$ for any flux function. In pulling the divergence out of the flux surface average we have used $\langle\nabla\cdot\mathbf{P}\rangle = \partial/\partial V\left\langle P^V\right\rangle$. We combine the first two divergence terms to find

$$\frac{\partial}{\partial V}\frac{A_\varphi}{c}\frac{\partial}{\partial t}\left\langle P^V\right\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle = \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle \frac{\partial P^V}{\partial t} + fe\dot{V}\right\rangle = 0 \tag{78}$$

which vanishes by Eqs. (64,65). Hence we have

$$\frac{A_\varphi}{c}\frac{\partial}{\partial t}\langle fe\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle = -\frac{\partial}{\partial t}\left\langle \frac{1}{c}\mathbf{P}\cdot\nabla A_\varphi\right\rangle \tag{79}$$

as a result for any H subject to the comment after Eq. (58), through the dependence of the vorticity equation (Eq. 64) upon the polarisation equation (Eq. 16). Insertion of Eq. (79) into Eq. (75) and moving the right side term to the left side produces

$$-\frac{\partial}{\partial t}\left\langle \frac{1}{c}\mathbf{P}\cdot\nabla A_\varphi\right\rangle + \frac{\partial}{\partial t}\langle fp_z b_\varphi\rangle + \frac{\partial}{\partial V}\left\langle fp_z b_\varphi \dot{V}\right\rangle + \left\langle f\frac{\partial H}{\partial \varphi}\right\rangle = 0 \tag{80}$$

This result eliminates terms in which A_φ, the flux label, appears by itself (not under a gradient operator). We may identify the first and second terms as the time derivative of a total toroidal momentum density consisting of an ExB part and a parallel part (both have toroidal components). The next term gives the drift effects of the parallel part of the toroidal momentum density. The last term has already been seen to vanish under the total phase space integral, so as a flux surface average it should be another (set of) transport divergence term(s). It remains to show this.

A. Field Terms for Simple Hamiltonian Dependences

We assume at first for illustration purposes an electrostatic case with general dependence of H upon ϕ and $\nabla\phi$ irrespective of any ordering. We have the integrand in the last term of Eq. (75) as

$$f\frac{\partial H}{\partial \varphi} = f\frac{\partial H}{\partial \phi}\frac{\partial \phi}{\partial \varphi} + f\frac{\partial H}{\partial \nabla\phi}\cdot\nabla\frac{\partial \phi}{\partial \varphi} \tag{81}$$

noting that the partial derivative and the gradient commute. The gradient is done by parts to find

$$f\frac{\partial H}{\partial \varphi} = \left(f\frac{\partial H}{\partial \phi} - \nabla\cdot f\frac{\partial H}{\partial \nabla\phi}\right)\frac{\partial \phi}{\partial \varphi} + \nabla\cdot\left(\frac{\partial \phi}{\partial \varphi}f\frac{\partial H}{\partial \nabla\phi}\right) \tag{82}$$

The first expression in parentheses is the functional derivative and it vanishes under the eventual species-summed velocity space integration, due to Eq. (16). The second term gives a divergence of a transport flux. Putting this back under the flux surface average, we have

$$\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle = \frac{\partial}{\partial V}\left\langle\frac{\partial \phi}{\partial \varphi}\nabla V\cdot f\frac{\partial H}{\partial \nabla\phi}\right\rangle \tag{83}$$

which is the desired result.

Following the same analysis for a general dependence of H upon ϕ and $A_\parallel$, with the functional derivative with respect to $A_\parallel$ not vanishing but replaced by the field term as per Eq. (17) we find

$$\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle = \frac{\partial}{\partial V}\left\langle\frac{\partial \phi}{\partial \varphi}\nabla V\cdot f\frac{\partial H}{\partial \nabla\phi}\right\rangle + \frac{\partial}{\partial V}\left\langle\frac{\partial A_\parallel}{\partial \varphi}\nabla V\cdot\left(f\frac{\partial H}{\partial \nabla A_\parallel} + \frac{1}{4\pi}\nabla_\perp A_\parallel\right)\right\rangle \tag{84}$$

where the contribution due to $\partial B_\perp^2/\partial\varphi$ vanishes under the flux surface average. As we will see when explicitly showing the version of the result using the long-wavelength "MHD Hamiltonian" these two terms generally give the Reynolds and Maxwell stresses, respectively.

B. Field Terms for Hamiltonian Dependences involving the Laplacian

In the case that H depends on the field variables through not only ϕ and $\nabla\phi$ but also $\nabla_\perp^2\phi$ then there is more to do but generalisation is straightforward (for background see the text by Gelfand and Fomin [48]). We expand

$$f\frac{\partial H}{\partial \varphi} = f\frac{\partial H}{\partial \phi}\frac{\partial \phi}{\partial \varphi} + f\frac{\partial H}{\partial \nabla\phi}\cdot\nabla\frac{\partial \phi}{\partial \varphi} + f\frac{\partial H}{\partial \nabla_\perp^2\phi}\nabla_\perp^2\frac{\partial \phi}{\partial \varphi} \tag{85}$$

again assuming the differential operator commutators vanish. The first two terms are done as before but the perpendicular Laplacian involves two integrations by parts. The Laplacian piece becomes

$$f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\nabla_\perp^2 \frac{\partial \phi}{\partial \varphi} = \nabla \cdot f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\nabla_\perp \frac{\partial \phi}{\partial \varphi} - \nabla_\perp \frac{\partial \phi}{\partial \varphi} \cdot \nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi} \tag{86}$$

Both of these pieces further expand according to

$$\nabla \cdot f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\nabla_\perp \frac{\partial \phi}{\partial \varphi} = \nabla \cdot \nabla_\perp \left(\frac{\partial \phi}{\partial \varphi} f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) - \nabla \cdot \left(\frac{\partial \phi}{\partial \varphi}\nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) \tag{87}$$

and

$$-\nabla_\perp \frac{\partial \phi}{\partial \varphi} \cdot \nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi} = -\nabla \cdot \left(\frac{\partial \phi}{\partial \varphi}\nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) + \frac{\partial \phi}{\partial \varphi}\nabla_\perp^2 f\frac{\partial H}{\partial \nabla_\perp^2 \phi} \tag{88}$$

The Laplacian piece is then

$$\begin{aligned} f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\nabla_\perp^2 \frac{\partial \phi}{\partial \varphi} &= \frac{\partial \phi}{\partial \varphi}\nabla_\perp^2 f\frac{\partial H}{\partial \nabla_\perp^2 \phi} \\ &+ \nabla \cdot \left[\nabla_\perp \left(\frac{\partial \phi}{\partial \varphi} f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) - 2\left(\frac{\partial \phi}{\partial \varphi}\nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right)\right] \end{aligned} \tag{89}$$

Putting in the terms from the ϕ and $\nabla\phi$ dependences, we have

$$\begin{aligned} f\frac{\partial H}{\partial \varphi} &= \frac{\partial \phi}{\partial \varphi}\left(f\frac{\partial H}{\partial \phi} - \nabla \cdot f\frac{\partial H}{\partial \nabla \phi} + \nabla_\perp^2 f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) \\ &+ \nabla \cdot \left[\frac{\partial \phi}{\partial \varphi}\left(f\frac{\partial H}{\partial \nabla \phi} - 2\nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) + \nabla_\perp \left(\frac{\partial \phi}{\partial \xi} f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right)\right] \end{aligned} \tag{90}$$

The first term in parentheses gives the functional derivative (which vanishes according to Eq. 16) and the rest become transport fluxes. The flux surface average with species-summed velocity space integration then yields

$$\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle = \frac{\partial}{\partial V}\left\langle \nabla V \cdot \left[\frac{\partial \phi}{\partial \varphi}\left(f\frac{\partial H}{\partial \nabla \phi} - 2\nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) + \nabla_\perp \left(\frac{\partial \phi}{\partial \varphi} f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right)\right]\right\rangle \tag{91}$$

which is the desired result.

Following the same analysis for a general dependence of H upon ϕ and $A_\parallel$, with the functional derivative with respect to $A_\parallel$ not vanishing but replaced by the field term as per Eq. (17) we find

$$\begin{aligned} \left\langle f\frac{\partial H}{\partial \varphi}\right\rangle &= \frac{\partial}{\partial V}\left\langle \nabla V \cdot \left[\frac{\partial \phi}{\partial \varphi}\left(f\frac{\partial H}{\partial \nabla \phi} - 2\nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right) + \nabla_\perp \left(\frac{\partial \phi}{\partial \varphi} f\frac{\partial H}{\partial \nabla_\perp^2 \phi}\right)\right]\right\rangle \\ &+ \frac{\partial}{\partial V}\left\langle \nabla V \cdot \left[\frac{\partial A_\parallel}{\partial \varphi}\left(\frac{1}{4\pi}\nabla_\perp A_\parallel + f\frac{\partial H}{\partial \nabla A_\parallel} - 2\nabla_\perp f\frac{\partial H}{\partial \nabla_\perp^2 A_\parallel}\right)\right.\right. \\ &\left.\left. + \nabla_\perp \left(\frac{\partial A_\parallel}{\partial \varphi} f\frac{\partial H}{\partial \nabla_\perp^2 A_\parallel}\right)\right]\right\rangle \end{aligned} \tag{92}$$

where the contribution due to $\partial B_\perp^2/\partial\varphi$ vanishes under the flux surface average.

Though more complicated, this result shows that for general dependence of H upon ϕ and $A_\parallel$ and their gradients and Laplacians, all terms in the toroidal momentum transport equation may be recast as divergences of flux surface-averaged transport fluxes, and no terms with A_φ not under gradients appear.

C. Illustration using an MHD Hamiltonian

We may dispense with finite gyroradius effects while studying equilibrium flow dynamics in the long wavelength regime (typically for edge equilibrium flows the local ρ_* is between 1/100 and 1/30) and also expecting $\nabla_\perp^2\phi > (1/n_e e)\nabla_\perp^2 p_i$ unless the flows are weak [27]. In conventional tokamaks the low-frequency and low-beta assumptions referred to above (Eq. 14) are well satisfied and the relevant form of MHD is reduced MHD [34, 35]. In general (cf. Refs. [7, 11]) there are contributions to polarisation by $A_\parallel$ but in this limit the terms due to $A_\parallel$ in $H_2 f$ are small compared to $B_\perp^2/8\pi$, so for these purposes it is sufficient to keep $A_\parallel$ only in the parallel kinetic energy term $mU^2/2$ and in $B_\perp^2/8\pi$. These considerations are interesting in their own right and will be treated in a different work. It is presently more important, however, to concentrate on the structure of the theory rather than the details of any particular version, so here we use the one necessary to obtain familiar reduced MHD forms. One further consideration deserves emphasis: no equation beyond the original statement of L and H requires justification; only L and H themselves. Once L and H are chosen, all assumption stops and further results are a matter of derivation.

The Hamiltonian used in Ref. [27] is

$$H = m\frac{U^2}{2} + \mu B + e\phi - \frac{1}{2}mv_E^2 \tag{93}$$

where the square of the ExB velocity and the parallel velocity are

$$v_E^2 = \frac{c^2}{B^2}\left|\nabla_\perp\phi\right|^2 \qquad mU = p_z - \frac{e}{c}A_\parallel \qquad \frac{\partial H}{\partial p_z} = U \tag{94}$$

For the derivatives of H we have

$$\frac{\partial H}{\partial \phi} = e \qquad \frac{\partial H}{\partial \nabla\phi} = -\frac{mc^2}{B^2}\nabla_\perp\phi \qquad \frac{\partial H}{\partial A_\parallel} = -\frac{e}{c}U \qquad \frac{\partial H}{\partial \nabla A_\parallel} = 0 \tag{95}$$

and there is no dependence on the Laplacian of either of the field potentials. The functional derivatives are

$$\frac{\delta fH}{\delta\phi} = \int d\mathcal{W}\left[fe + \frac{1}{B_\parallel^*}\nabla\cdot B_\parallel^*\frac{fmc^2}{B^2}\nabla_\perp\phi\right] \qquad \frac{\delta fH}{\delta A_\parallel} = -\int d\mathcal{W}\,\frac{e}{c}fU \tag{96}$$

The polarisation vector is

$$\mathbf{P} = -\rho_M\frac{c^2}{B^2}\nabla_\perp\phi \tag{97}$$

Inserting these forms into Eq. (80) using Eq. (84), we find upon collecting the ∇V terms

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle\rho_M\frac{c}{B^2}\nabla\phi\cdot\nabla A_\varphi + fp_z b_\varphi\right\rangle \\ &\quad+\frac{\partial}{\partial V}\left\langle\nabla V\cdot\left(fp_z b_\varphi\dot{\mathbf{R}} - \rho_M\frac{c^2}{B^2}\frac{\partial\phi}{\partial\varphi}\nabla\phi + \frac{1}{4\pi}\frac{\partial A_\parallel}{\partial\varphi}\nabla A_\parallel\right)\right\rangle = 0\end{aligned} \tag{98}$$

which is the desired result. We have the conserved local toroidal momentum density, the quantity under the time derivative. The pieces are the covariant toroidal angle components of the ExB and parallel momenta, respectively. The quantity under the divergence is the radial flux of the toroidal momentum. The pieces are the magnetic flutter and ExB/parallel Reynolds stress, the pure ExB Reynolds stress, and the Maxwell stress. This is the result as obtained without the use of any ordering except that involved in the prescription of L.

We also consider this as a mean field theory, in which the flux surface average is also understood to contain a time average over a mesoscale range longer than eddy correlation times but shorter than transport diffusion times. The fluctuations are assumed to be small amplitude (order ρ_* in relative amplitude, with velocities normalised to the sound speed). Following the considerations in Sec. VI, we apply flute mode ordering to the fluctuations, with the observed relationship between φ and ξ in covariant components. Turbulent fluxes appear in the flux surface averages where the V-component of equilibrium flows is negligible (the magnetic drifts constitute neoclassical transport, which we neglect here). In the transported quantity the contribution due to the fluctuations is neglected. See Ref. [38] for application of this in a drift-fluid model. We will also take the single-fluid MHD approximations, where each species is assumed to have the same parallel velocity and in the end there is a single pressure. In this section, the tilde symbol denotes fluctuations in the indicated quantities.

To see the ExB toroidal momentum we note that

$$\frac{c}{B^2}\nabla\phi\cdot\nabla A_\varphi = \frac{c}{B^2}\nabla\phi\cdot[\nabla\varphi\times(\nabla A_\varphi\times\nabla\varphi)] = R^2\frac{c}{B^2}\mathbf{B}\times\nabla\phi\cdot\nabla\varphi = v_\vartheta \tag{99}$$

using Eq. (66) and $R^2 |\nabla\varphi|^2 = 1$ with R the toroidal major radius, and noting that the toroidal magnetic field does not contribute. Thus we may combine

$$\left\langle \rho_M \frac{c}{B^2} \nabla\phi \cdot \nabla A_\varphi + f p_z b_\varphi \right\rangle = \langle \rho_M u_\varphi \rangle \tag{100}$$

into a toroidal momentum density given by the mass density times the toroidal flow u_φ under the mean field and single-fluid MHD approximations.

For the Reynolds and Maxwell stresses, we observe that

$$\frac{c}{B^2} \nabla V \cdot \nabla \widetilde{\phi} = \frac{c}{\chi'^2} \left(\frac{g^{VV}}{g^{VV} g^{\xi\xi} - g^{V\xi} g^{\xi V}} \frac{\partial \widetilde{\phi}}{\partial V} + \frac{g^{V\xi}}{g^{VV} g^{\xi\xi} - g^{V\xi} g^{\xi V}} \frac{\partial \widetilde{\phi}}{\partial \xi} \right) \tag{101}$$

Since $F^{\xi V}/B^2 = 1/\chi'$ the ExB velocity components are

$$\widetilde{v}^V = \frac{c}{\chi'} \frac{\partial \widetilde{\phi}}{\partial \xi} \qquad\qquad \widetilde{v}^\xi = -\frac{c}{\chi'} \frac{\partial \widetilde{\phi}}{\partial V} \tag{102}$$

The covariant metric coefficients are

$$g_{\xi\xi} = \frac{g^{VV}}{g^{VV} g^{\xi\xi} - g^{V\xi} g^{\xi V}} \qquad\qquad g_{\xi V} = \frac{-g^{V\xi}}{g^{VV} g^{\xi\xi} - g^{V\xi} g^{\xi V}} \tag{103}$$

Using the index lowering operation

$$\widetilde{v}_\xi = g_{\xi\xi} \widetilde{v}^\xi + g_{\xi V} \widetilde{v}^V \tag{104}$$

we have

$$-\frac{c^2}{B^2} \frac{\partial \widetilde{\phi}}{\partial \xi} \nabla V \cdot \nabla \widetilde{\phi} = \widetilde{v}^V \widetilde{v}_\xi \tag{105}$$

and putting the 2π normalisation back in, we have

$$-\frac{c^2}{B^2} \frac{\partial \widetilde{\phi}}{\partial \varphi} \nabla V \cdot \nabla \widetilde{\phi} = \widetilde{v}^V \widetilde{v}_\vartheta \tag{106}$$

which is the radial/toroidal Reynolds stress. With magnetic fluctuations given by

$$\widetilde{b}^V = -\frac{1}{\chi'} \frac{\partial \widetilde{A}_\parallel}{\partial \xi} \qquad\qquad \widetilde{b}^\xi = \frac{1}{\chi'} \frac{\partial \widetilde{A}_\parallel}{\partial V} \tag{107}$$

we similarly have

$$\frac{1}{4\pi} \frac{\partial \widetilde{A}_\parallel}{\partial \varphi} \nabla V \cdot \nabla \widetilde{A}_\parallel = -\frac{B^2}{4\pi} \widetilde{b}^V \widetilde{b}_\varphi = -\widetilde{B}^V \widetilde{B}_\varphi \tag{108}$$

which is the radial/toroidal Maxwell stress.

For the magnetic flutter nonlinearity and the ExB/parallel Reynolds stress we note that the part of the drifts due to fluctuations is given by

$$\dot{\mathbf{R}} = \nabla\widetilde{H} \cdot \frac{c\mathbf{F}}{eB^2} = \nabla\widetilde{\phi} \cdot \frac{c\mathbf{F}}{B^2} - U\nabla\widetilde{A}_\parallel \cdot \frac{\mathbf{F}}{B^2} \tag{109}$$

where to lowest order in ρ_* we approximate $B^*_\parallel \to B$ and neglect the second order drift term due to $-mv_E^2/2$ in H. The V-component of this is

$$\dot{V} = \frac{c}{\chi'}\left(\frac{\partial\widetilde{\phi}}{\partial\xi} - U\frac{\partial\widetilde{A}_\parallel}{\partial\xi}\right) \tag{110}$$

The flux surface average of the fluctuation drifts term is then

$$\left\langle \widetilde{f} p_z b_\varphi \dot{V} \right\rangle \to \left\langle \widetilde{f} p_z b_\varphi \frac{c}{\chi'}\frac{\partial\widetilde{\phi}}{\partial\xi} - fUp_z b_\varphi \frac{c}{\chi'}\frac{\partial\widetilde{A}_\parallel}{\partial\xi} \right\rangle \tag{111}$$

These are two different effects in a gyrofluid sense because they involve different moments of f. In both terms $f \to \widetilde{f}$ represents fluctuations because a first-order fluctuation term vanishes under the time average contained in $\langle\,\rangle$. Considering this, we have the moments

$$\left\langle nm\widetilde{u}_\parallel \widetilde{v}^V b_\varphi + \widetilde{p}_\parallel \widetilde{b}^V b_\varphi \right\rangle \tag{112}$$

where $\widetilde{v}^V$ and $\widetilde{b}^V$ are as given in Eqs. (102,107). These two terms give the ExB/parallel Reynolds stress and the magnetic flutter transport pieces. The distinctions between U and p_z/m and between $p_\parallel$ and $P_\parallel = p_\parallel + nmu_\parallel^2$ are neglected for small amplitude fluctuations.

At the MHD level the sum over species leads back to the total mass density ρ_M and the total pressure p which replace nm and $p_\parallel$ (*i.e.*, neglecting anisotropy), respectively. This leads to

$$\frac{\partial}{\partial t}\left\langle \rho_M u_\varphi \right\rangle + \frac{\partial}{\partial V}\left\langle \rho_M \widetilde{v}^V \widetilde{v}_\vartheta + \rho_M \widetilde{u}_\parallel \widetilde{v}^V b_\varphi - \frac{1}{4\pi}\widetilde{B}^V \widetilde{B}_\varphi + \widetilde{p}\widetilde{b}^V b_\varphi \right\rangle = 0 \tag{113}$$

under MHD and mean field approximations having used the "MHD Hamiltonian" given in Eq. (93). We have placed the two Reynolds stress terms next to each other and left them separate, for clarity. All of these terms are easily identifiable with well known processes within the MHD fluid model. We note that the mean field ordering has been used for evaluation purposes only, with the actual transport equation for this model given by Eq. (98).

This exercise has served to prove that the general gyrokinetic toroidal momentum conservation laws can be brought back to MHD via straightforward application of the MHD approximations.

VIII. THE ENERGY TRANSPORT EQUATION

The energy transport equation is much better known, even for gyrokinetic theory. However there is an important manipulation concerning the part of H due to ϕ. Decomposing $H = H_0 + e\phi + H_P$ as in Sec. V, we start with

$$fH = fH_0 + fe\phi + fH_P \tag{114}$$

Summing over species and inserting Eqs. (54,55) we find

$$\sum_{\text{sp}} \int d\mathcal{W}\, fH = \sum_{\text{sp}} \int d\mathcal{W}\, f(H_0 + H_P) + \nabla \cdot \phi \mathbf{P} - \mathbf{P} \cdot \nabla \phi \tag{115}$$

and under the flux surface average

$$\langle fH \rangle = \langle f(H_0 + H_P) \rangle - \langle \mathbf{P} \cdot \nabla \phi \rangle + \frac{\partial}{\partial V} \left\langle \phi P^V \right\rangle \tag{116}$$

We also have

$$\left\langle fH\dot{V} \right\rangle = \left\langle [f(H_0 + H_P) - \mathbf{P} \cdot \nabla \phi]\, \dot{V} \right\rangle + \left\langle (\nabla \cdot \phi \mathbf{P})\, \dot{V} \right\rangle \tag{117}$$

where as before $\dot{V} = \dot{\mathbf{R}} \cdot \nabla V$. Reduction of the time derivative term in Eq. (51) is done the same way as in Sec. VII for the toroidal angle derivative term in Eq. (50), noting that the time derivative of $B_\perp^2$ survives, as in the analysis leading to Eq. (33). We find

$$\begin{aligned}
&\frac{\partial}{\partial t} \left\langle fH_0 + fH_P - \langle \mathbf{P} \cdot \nabla \phi \rangle + \frac{B_\perp^2}{8\pi} \right\rangle + \frac{\partial}{\partial V} \left\langle fH_0 \dot{V} \right\rangle \\
&\quad + \frac{\partial}{\partial V} \left\langle (fH_P - \langle \mathbf{P} \cdot \nabla \phi \rangle + \nabla \cdot \phi \mathbf{P})\, \dot{V} + \frac{\partial}{\partial t} \phi P^V \right\rangle \\
&\quad - \frac{\partial}{\partial V} \left\langle \nabla V \cdot \left[\frac{\partial \phi}{\partial t} \left(f \frac{\partial H}{\partial \nabla \phi} - 2 \nabla_\perp f \frac{\partial H}{\partial \nabla_\perp^2 \phi} \right) + \nabla_\perp \left(\frac{\partial \phi}{\partial t} f \frac{\partial H}{\partial \nabla_\perp^2 \phi} \right) \right] \right\rangle \\
&\quad - \frac{\partial}{\partial V} \left\langle \nabla V \cdot \left[\frac{\partial A_\parallel}{\partial t} \left(\frac{1}{4\pi} \nabla_\perp A_\parallel + f \frac{\partial H}{\partial \nabla A_\parallel} - 2 \nabla_\perp f \frac{\partial H}{\partial \nabla_\perp^2 A_\parallel} \right) \right. \right. \\
&\qquad \left. \left. + \nabla_\perp \left(\frac{\partial A_\parallel}{\partial t} f \frac{\partial H}{\partial \nabla_\perp^2 A_\parallel} \right) \right] \right\rangle \\
&\qquad = 0
\end{aligned} \tag{118}$$

In practical cases only the terms on the top line are significant. Under the time derivative we have the thermal and kinetic energy, ExB energy, and magnetic energy. The transport term gives the ExB advection (the $\nabla\phi$ term in H in the drifts) and magnetic flutter (the $U\nabla A_\parallel$ term in H in the drifts). All the others are polarisation and induction corrections. Inserting

the MHD Lagrangian from Sec. VII C, and taking the single-fluid MHD approximations, we find

$$\langle fH_0 \rangle = \frac{3}{2} \langle p \rangle + \frac{1}{2} \left\langle \rho_M u_\parallel^2 \right\rangle \qquad \langle fH_P - \mathbf{P} \cdot \nabla \phi \rangle = \frac{1}{2} \left\langle \rho_M v_E^2 \right\rangle \tag{119}$$

together with $B_\perp^2/8\pi$ for the energy pieces, and

$$\left\langle fH_0 \dot{V} \right\rangle = \frac{3}{2} \left\langle \widetilde{p} \widetilde{v}^V \right\rangle + \left\langle q_{e\parallel} \widetilde{b}^V \right\rangle \tag{120}$$

where $q_{e\parallel}$ is the conductive electron parallel heat flux, for the dominant transport pieces. The correspondence with MHD forms is evident. This exercise has served to prove that the general gyrokinetic energy conservation laws can be brought back to MHD via straightforward application of the MHD approximations.

IX. SUMMARY AND DISCUSSION

The main results of this work are the antisymmetric 4-bracket form of the gyrokinetic equation, Eq. (24), the global conservation laws for energy and toroidal momentum, Eqs. (33,48), the local phase space advection equations for energy and toroidal momentum, Eqs. (28,41), the phase space continuity equations for energy and toroidal momentum, Eqs. (50,51), and the transport equations for vorticity, toroidal momentum, and energy, Eqs. (63,80,118) with Eqs. (84,92) as auxiliaries. It is important to note that no ordering assumptions were required to obtain these. The only required conditions are the form of the Lagrangian and Hamiltonian implied by Eqs. (1,2,13), the ability to write the vorticity in the form of a polarisation vector divergence as in Eqs. (54,55), which enable the cancellation given in Eq. (79). This in turn leads to the single condition on the form of H, that the dependence upon ϕ must be as in Eq. (53), which leads to Eqs. (54,55,58). These results confirm what Refs. [12, 13] already implied and Ref. [11] already reviewed. The part of the results that are novel comprises the 4-bracket form of the gyrokinetic equation, its use in proving the conservation laws, and that the local forms of the conservation laws, *i.e.*, the transport equations, have a solid fundamental basis for any reasonable choice of Lagrangian. The correspondence to nonlinear reduced MHD was shown, also without ordering, in Eq. (98).

The usefulness and significance of the ability to arrange the Lie transforms with which L_p and H are built such that all time and toroidal angle dependence is kept in the dynamical

field variables and the latter are strictly contained in H (*i.e.*, to obtain the form of Eq. 1). Previous work has already shown how to arrange this, for both the conventional small scale and also for the newer large scale orderings [6, 9].

The role of the cancellation in Eq. (79) highlights the role of the vorticity equation in the overall consideration of momentum. In a low-frequency fluid drift model, the natural decomposition of rotation is not poloidal/toroidal but perp/parallel, with the ExB energy equation controlling the evolution of the perpendicular flow. In a gyrokinetic or gyrofluid model, this role is taken over by the ion gyrocenter density variable, the energy content is controlled by fH_E (where H_E is the part of H which depends on ϕ), and the relevant conserved quantity is, one and the same, gyrocenter (apparent) charge density and generalised vorticity. The appearance of the polarisation density as the divergence of a polarisation vector is a fundamental property underlying the conservation laws. Hence, the local conservation of toroidal momentum depends on the simultaneous conservation of vorticity. At the level of the equations, obtaining the one is dependent on the use of the other to provide a cancellation by which the amplitude of the poloidal magnetic flux (as opposed to its gradient) is removed from influence on the transport of toroidal momentum.

The mathematical functioning of the global conservation equations for energy and toroidal momentum depends critically on the ability to employ time and toroidal angle translation symmetry, respectively, in the use of functional derivatives to evaluate an apparent residual which in fact vanishes under the integral (the right hand sides of Eqs. 28,41). This in turn depends on the exact working of the Euler-Lagrange equations for the field variables, and this in turn is defined in terms of functional derivatives.

At the level of the local equations, these apparent residuals are shown to be recastable in terms of divergences using techniques similar to the use of functional derivatives. Hence, for any H, they are recast as transport fluxes. All of the transport fluxes which remain significant under mean field and small fluctuation ordering are found to have clear MHD analogues, another step in confirming the general solidity of the theory. Using a simplified Hamiltonian defined in Eq. (93) which keeps finite gyroradius residual corrections only in the retention of the ExB energy as a second order drift, it was shown to be straightforward to recover the MHD limit of the transport equations (Eq. 98), with the familiar Reynolds and Maxwell stresses and the magnetic flutter nonlinearity elucidated by a mean field analysis (Eq. 113). This "MHD Hamiltonian" provides the link between gyrokinetics and nonlinear

reduced MHD at the total-f level. The correspondence at the delta-f level was recently proved elsewhere for gyrofluid theory [49].

A subsidiary analysis was used to show that for small scale fluctuation ordering, the magnetic flux term (A_φ/c), which is formally of order ϵ^{-1} as discussed in Ref. [8] does not introduce terms in H at higher order than already are necessary to evaluate the momentum transport equation. This result, in Eq. (72), is sufficient to allay recent concerns about the integrity of the treatment of momentum conservation and transport by gyrokinetics which have been voiced by others [31].

Second order drifts are necessary for energetic consistency if polarisation is nonlinear, but it is possible to have a model with linearised polarisation and only first order drifts in the gyrokinetic equation itself and still satisfy energetic consistency. Such models also have well behaved local transport equations for toroidal momentum, as shown in Appendix B 1. The second order Hamiltonian terms — quadratic in the field dependent variables — must be present in some form, either as second order drifts in the gyrokinetic equation or as background field terms, since they control the dynamical energy accounted for by those variables.

Energetic consistency underlies all the derivations, which do not work otherwise. Any model which does not have this at its heart and which attempts to go beyond first order field dependence in drift terms is prescribed to fail on consistency grounds. It is important to note that the abovementioned concerns were not done with an energetically consistent analysis and did not address any of the previous results on energetic consistency nor even the existence of gyrokinetic field theory. However, the seminal references on gyrokinetic field theory already clearly demonstrated the need to keep fully nonlinear polarisation at any level of ordering past first order dependence of H upon the field variables [12, 13], and this was and is followed rigorously in field theory treatments of reduced (drift) fluid equations [50, 51] and total-f gyrofluid equations [36, 37].

Gyrokinetic field theory was not necessary to build the original gyrokinetic computational models [28–30]. However, an equivalent field theory can be built from the given gyrokinetic Poisson equation (multiply by the time derivative of ϕ and find the resulting form with f times the time derivative of H, then use that H to re-derive the equations). The above-cited computations and more recent ones derived from them [52–56] are consistent with this. This is why conventional delta-f global computation is on solid fundamentals.

Gyrokinetic field theory is however the only form of modern gyrokinetic theory as it is required to provide well founded generalisations in practically any context. It is also necessary to any effort which aims to generalise gyrokinetic theory in practically any way.

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Appendix A: Use of Functional Derivatives

The Euler-Lagrange equations for the fields are found by identifying the functional derivatives of L with respect to each field potential and setting them to zero. For an introduction to the use of functional derivatives in this type of field theory see Section II B of the review by Morrison [57]. For background on functional derivatives within the general context of the calculus of variations see the text by Gelfand and Fomin [48]. For the more familiar version by which stress tensor theorems are proved within classical field theory see the text by Landau and Lifshitz [32].

The functional derivatives are defined with respect to the space covered by $d\mathcal{V}$, so that, e.g., variation of ϕ results in

$$\delta L(\delta\phi) \equiv \int d\mathcal{V}\, \delta\phi \, \frac{\delta L}{\delta \phi} \tag{A1}$$

in which it is understood that the definition of the functional derivative includes velocity space integration. The arbitrariness of $\delta\phi$ and the extremal requirement for L give the field

equation as

$$\frac{\delta L}{\delta \phi} = 0 \tag{A2}$$

Since the dependence of L on ϕ is through H only then this is equivalent to

$$\sum_{\text{sp}} \frac{\delta f H}{\delta \phi} = 0 \tag{A3}$$

Variation of $A_\parallel$ results in

$$\delta L(\delta A_\parallel) \equiv \int d\mathcal{V}\, \delta A_\parallel \frac{\delta L}{\delta A_\parallel} \tag{A4}$$

and the arbitrariness of $\delta A_\parallel$ and the extremal requirement for L give the field equation as

$$\frac{\delta L}{\delta A_\parallel} = 0 \tag{A5}$$

Since the dependence of L on $A_\parallel$ includes the field term $-B_\perp^2/8\pi$ this is equivalent to

$$\sum_{\text{sp}} \frac{\delta f H}{\delta A_\parallel} = \frac{1}{4\pi} \nabla_\perp^2 A_\parallel \tag{A6}$$

These are the field equations for arbitrary H given the field term $B_\perp^2/8\pi$. The induction equation depends on the form of this term; for example the version of $B_\perp^2$ which gives a result closer to the form of Ampère's law used in the Grad-Shafranov equation for MHD equilibrium is

$$B_\perp^2 = \frac{1}{R^2} \left| \nabla_\perp (A_\parallel R) \right|^2 \tag{A7}$$

where R is the toroidal major radius. In this case the field equation is given by

$$\sum_{\text{sp}} \frac{\delta f H}{\delta A_\parallel} = \frac{R}{4\pi} \nabla \cdot \frac{1}{R^2} \nabla_\perp (A_\parallel R) \tag{A8}$$

with the new form on the right side. Due to the axisymmetry of R, either of these forms may be used in the above derivations.

1. Integrals over f and H

The conservation laws depend on phase space integrals over expressions of f and H in the form

$$\sum_{\text{sp}} \int d\Lambda\, f\, d_a\, H \tag{A9}$$

where d_a is a differential operator with the linearity property, (e.g., a time derivative). In turn, the dependence of H on the fields involves the field amplitudes and their derivatives which we can schematically express as

$$H = H(\phi, d_b\phi) \tag{A10}$$

where d_b is another differential operator with either the linearity or the the Hermitian property (e.g., ∇ or $\nabla_\perp^2$ or J_0). The relevant differential of H is

$$d_a H = \frac{\partial H}{\partial \phi} d_a\phi + \frac{\partial H}{\partial d_b\phi} d_a d_b \phi \tag{A11}$$

We can turn the integral into a functional derivative form if and only if the two operators commute:

$$\int d\Lambda\, f\, d_a\, H = \int d\Lambda\, \frac{\delta f H}{\delta \phi} d_a\phi \qquad \Longleftrightarrow \qquad (d_a d_b - d_b d_a)\phi = 0 \tag{A12}$$

If all the d_b in H commute with $\partial/\partial t$ then we have time symmetry which is a prerequisite for energy conservation. If all the d_b in H commute with $\partial/\partial\varphi$ then we have toroidal angle (axi-)symmetry which is a prerequisite for toroidal momentum conservation. In general the commutator does not vanish if d_a is ∇, since d_b involves B which is spatially dependent. In stellarator geometry $\partial B/\partial\varphi \neq 0$ and therefore toroidal momentum conservation is not conserved (it is exchanged with $\mathbf{B}$). In tokamak geometry poloidal momentum is exchanged with $\mathbf{B}$ but toroidal momentum is conserved in the ideal case (no destruction of axisymmetry).

Appendix B: Equivalent Hamiltonians to Conventional Models

As an example of how one can back-construct a gyrokinetic field theory from a conventional model, we consider the gyrokinetic Poisson equation as currently used in most numerical simulations [52–54, 56]. The gyrocenter charge density piece is kept with gyroaveraging and the polarisation is given by a background ion density,

$$\sum_{\rm sp} \int d\mathcal{W}\, [eJ_0 f] + \nabla \cdot \frac{n_0 M_i c^2}{B^2} \nabla_\perp \phi = 0 \tag{B1}$$

For this to come from a Lagrangian it has to be the functional derivative with respect to ϕ. We may rebuild this by multiplying by $\delta\phi$ and integrating over the volume, so that

$$\sum_{\rm sp} \int d\Lambda\, \delta\phi\, [eJ_0 f] + \int d\mathcal{V}\, \delta\phi\, \nabla \cdot \frac{n_0 M_i c^2}{B^2} \nabla_\perp \phi = 0 \tag{B2}$$

Here, all we need know about J_0 is that it is Hermitian. Doing the relevant integrations by parts we have

$$\sum_{\text{sp}} \int d\Lambda \left[f e J_0(\delta\phi) \right] - \int d\mathcal{V} \, \frac{n_0 M_i c^2}{B^2} \nabla_\perp \phi \cdot \nabla_\perp (\delta\phi) = 0 \tag{B3}$$

We identify the relevant parts of the Lagrangian as

$$L = \cdots - \sum_{\text{sp}} \int d\Lambda \, f(e J_0 \phi) + \int d\mathcal{V} \, \frac{n_0 M_i c^2}{2B^2} \left| \nabla_\perp \phi \right|^2 \tag{B4}$$

Variation of this with respect to ϕ indeed recovers Eq. (B1). Inspection of the version of the gyrokinetic equation used in these references shows that indeed $H_E = e J_0 \phi$ is used as the perturbed Hamiltonian in the drifts. Therefore, energetic consistency is assured.

Most versions of this model are considered with electrons in adiabatic force balance parallel to the magnetic field,

$$\int d\mathcal{W} f_e = n_0 \left[1 + \frac{e_0}{T_e} \left(\phi - \langle \phi \rangle \right) \right] \tag{B5}$$

with gyroaveraging neglected and with the same n_0 as in the background ExB energy term. The factor e_0 is the fundamental unit of charge, so written to distinguish from use of e as a species charge. The temperature T_e is a flux function and the subtraction of the flux surface average reflects the vanishing of the parallel gradient for this component. This state of force balance is expected to evolve adiabatically in local thermodynamic equilibrium, so the contribution to L is a profile anchor piece plus a fluctuation energy piece,

$$L_{\text{electrons}} = n_0 e_0 \phi + n_0 \frac{e_0^2}{2T_e} \left(\phi - \langle \phi \rangle \right)^2 \tag{B6}$$

which becomes a field term. With this substitution made in L, the polarisation equation is

$$n_0 \frac{e_0^2}{T_e} \left(\phi - \langle \phi \rangle \right) - \nabla \cdot \frac{n_0 M_i c^2}{B^2} \nabla_\perp \phi = -n_0 e_0 + \sum_{\text{ions}} \int d\mathcal{W} \, e J_0 f \tag{B7}$$

where the sum is over the ions only (for singly charged ions the same n_0 is used everywhere; for more species the appropriate adjustments are made to keep the profiles charge neutral). The term $n_0 e_0$ subtracts the profile piece from the ions. Energy conservation for this model was proved for the ORB code in Ref. [52]. The same model is also used by the GYSELA code [53].

The form given by Lee in the original gyrokinetic/Poisson system [28, 29, 58] is

$$\Psi = J_0 \phi - \frac{q}{2T} \frac{v_t^2}{\Omega_i^2} \left| \frac{\partial \phi(\mathbf{R})}{\partial \mathbf{R}} \right|^2 \tag{B8}$$

where Ψ is the potential with which particles are pushed in a gyrokinetic particle in cell model, ϕ is given strictly as a function of the gyrocenter position $\mathbf{R}$, the symbol used for gyroaveraging is replaced by our use of J_0, and q, T, and $(v_t/\Omega_i)^2 = mTc^2/q^2B^2$ are constant parameters, with q the same as our e. This is Eq. (2) of Ref. [29]. It is also an electrostatic model with $A_\parallel = 0$ and p_z/m interchangeable with a parallel velocity.

In our notation we write this as $H_E = e\Psi$ with

$$H_E = eJ_0\phi - \frac{mc^2}{2B^2}\left|\nabla\phi\right|^2 \tag{B9}$$

In other words, this is the same choice as our MHD Hamiltonian (Eq. 93), except for the use of the gyroaveraged ϕ as the first term and the terms dependent upon $A_\parallel$ in the MHD case. Lee's gyrokinetic Poisson equation is written in Eq. (3) of Ref. [29] as

$$\nabla^2\phi - \frac{\tau}{\lambda_D^2}(1-\Gamma_0)\phi + \frac{\rho_s^2}{\lambda_D^2}\nabla\cdot\frac{n_i-n_0}{n_0}\nabla_\perp\phi = -4\pi e_0(\overline{n}_i - n_e) \tag{B10}$$

The operator Γ_0 results from J_0^2 integrated against a Maxwellian and is given formally by multiplication in wavenumber space of Fourier coefficients by the $\Gamma_0(b) = e^{-b}I_0(b)$, where I_0 is the zeroth modified Bessel function, and the argument is $b = k_\perp^2\rho^2$ with ρ the species thermal gyroradius given by $\rho^2 = mTc^2/e^2B^2$. The factor $\overline{n}_i$ with the overbar denotes velocity space integration of $J_0 f_i$, the n_i without the overbar is velocity space integration of f_i, and $\tau/\lambda_D^2 = 4\pi n_0 e_0^2/T_i$ and $\rho_s^2/\lambda_D^2 = 4\pi n_0 M_i c^2/B_0^2$ and n_0 are constant parameters. Using the definitions in his Eqs. (4-7), this is found to be equivalent to

$$\frac{1}{4\pi}\nabla^2\phi + \sum_{\rm sp}\int d\mathcal{W}\left[F^M\frac{e^2}{T}(J_0^2 - 1 + \rho^2k_\perp^2)\phi + \frac{1}{B_\parallel^*}\nabla\cdot B_\parallel^*\frac{fmc^2}{B^2}\nabla_\perp\phi + eJ_0 f\right] = 0 \tag{B11}$$

where F^M is a Maxwellian with species parameters n and T, the correction factor including J_0^2 and $k_\perp^2\rho^2$ restores full finite gyroradius (FLR) effects to the field term, the sum over species using $m_e \to 0$ includes the electrons only in the last term $eJ_0 f$, and we have restored a toroidal model with the factors B and $B_\parallel^*$ arranged to preserve Hermicity of all the operators. The first term is true charge separation, the second with F^M is a field term, and the rest is the part due to the dependent variable. If we prescribe an electrostatic Lagrangian,

$$L = \sum_{\rm sp}\int d\Lambda\, f\, L_p + \sum_{\rm sp}\int d\Lambda\, F^M\left\{\frac{e^2}{2T}\left[\phi^2 - (J_0\phi)^2\right] - \frac{mc^2}{2B^2}\left|\nabla_\perp\phi\right|^2\right\} + \int d\mathcal{V}\,\frac{1}{8\pi}\left|\nabla\phi\right|^2 \tag{B12}$$

with the particle Lagrangian given by

$$L_p = (\mathbf{A} + p_z \mathbf{b}) \cdot \dot{\mathbf{R}} + \frac{mc}{e} \mu \dot{\vartheta} - \left(\frac{p_z^2}{2m} + \mu B + H_E \right) \tag{B13}$$

then the Euler-Lagrange equations recover gyrokinetic equations with $e\Psi$ as the ϕ-dependent part of H and the polarisation equation given by Lee as his gyrokinetic Poisson equation; in other words, Eqs. (1-3) of Ref. [29]. These were given for slab geometry but with the dependences of B placed as shown here together with the use of B_0 where indicated, this recovers a toroidal model with some extra field terms in L compared to ours. Since the field correction term in the polarisation equation is the functional derivative of a positive definite quantity, it still conserves energy. Since we are able to recast the field terms as a field Lagrangian, and to show otherwise that the particle pushing potential and the gyrocenter charge terms arise from the same term H_E in the Hamiltonian, energetic consistency is entirely satisfied.

Various simplified versions are given. For example, in the GTC code [59, 60] the particle pushing potential is consistent with $H_E = eJ_0\phi$ and the gyrokinetic Poisson equation is given as

$$\frac{\tau}{\lambda_D^2}(1 - \Gamma_0)\phi = 4\pi e_0(\overline{n}_i - n_e) \tag{B14}$$

then we have a linearised polarisation model consistent with

$$L = \sum_{\text{sp}} \int d\Lambda \, f \, L_p + \sum_{\text{sp}} \int d\Lambda \, F^M \frac{e^2}{2T} \left[\phi^2 - (J_0\phi)^2 \right] \tag{B15}$$

where L_p is as in Ref. (B13) with the simplified H_E, and adiabatic electrons are prescribed through Eq. (B5) above. Since using this L the above polarisation equation is recovered along with use of $\Psi = J_0\phi$ in the gyrocenter drifts, the model remains energetically consistent. Approximation of the $F^M(J_0^2 - 1)$ to long wavelength then recovers the ORB/GYSELA model mentioned above.

Lee mentions these models also in Ref. [28], giving their source in another model used to obtain his Eqs. (21,22). The particle pushing potential is given as

$$\Psi = J_0\phi + \frac{e}{2T} \left[(J_0\phi)^2 - J_0(\phi^2) \right] \tag{B16}$$

in our notation. The factor $1/T$ is obtained by approximating $\partial f/\partial\mu$ as $(-B/T)f$ since the second order term leads to the polarisability, which should be dominated by the largest scale

part of f which should not depart significantly from a Maxwellian. This Ψ was then used to recover the model given in Eqs. (B8,B10) by using the long wavelength approximation of the second order term. We write the interaction part, $fH_E = fe\Psi$, of the Lagrangian as

$$L = \cdots - \sum_{\rm sp} \int d\Lambda \, f \, H_E \tag{B17}$$

then undo this effective integration by parts under the phase space integration to obtain a similar model with

$$H_E = eJ_0\phi - \frac{e^2}{2B}\frac{\partial}{\partial\mu}\left[J_0(\phi^2) - (J_0\phi)^2\right] \tag{B18}$$

With this form the field term FLR correction is unnecessary and with quasineutrality we recover use of an L in which the only dependence upon ϕ is in H. The corresponding polarisation equation is

$$\sum_{\rm sp} \int d\mathcal{W} \left[eJ_0 f + \left(J_0\mathcal{M}J_0 - [J_0\mathcal{M}]\right)\phi\right] = 0 \tag{B19}$$

where the polarisability $\mathcal{M}$ is given by

$$\mathcal{M} = -\frac{e^2}{B}\frac{\partial f}{\partial\mu} \tag{B20}$$

It is very difficult to be able to take this derivative in a particle in cell model, which is why this second order term is usually approximated in some fashion.

This is very close to the version given by Hahm [6], who obtains an additional term from the second order part of the Lie transform so that

$$H_E = e\left\langle\phi\right\rangle_{\mathbf{R}} - \frac{e^2}{2B}\frac{\partial}{\partial\mu}\left\langle\widetilde{\phi}^2\right\rangle_{\mathbf{R}} - \frac{mc^2}{2B^2}\left\langle\nabla\widetilde{\Phi}\cdot\mathbf{b}\times\nabla\widetilde{\phi}\right\rangle_{\mathbf{R}} \tag{B21}$$

where in this case the subscripted angle brackets denote J_0 at the particle level and

$$\widetilde{\phi} = \phi - \left\langle\phi\right\rangle_{\mathbf{R}} \qquad\qquad \widetilde{\Phi} = \int^{\vartheta} \widetilde{\phi}\, d\vartheta \tag{B22}$$

The first of the two second order terms is equivalent to Lee's. The second involves an indefinite gyrophase integral and is difficult to compute. The equivalent polarisation equation is then given keeping all the relevant terms through use of the Lie back-transform. As noted above, this is equivalent to the use of field theory [11]. However, the added term in H_E yields a contribution to the polarisation equation which is one order down in ρ_* from the others, and it is never kept in computations. Nevertheless, with the approximations always effectively made in L and nowhere thereafter, exact energetic consistency is preserved in all of the versions. This effectively brings us back to Lee's H_E.

1. Momentum conservation with linearised polarisation

Here we demonstrate the existence of a familiar toroidal momentum conservation equation within the simplest possible form of a global gyrokinetic model. The dynamics is electrostatic, quasineutral, gyroaveraging is neglected, and polarisation is provided by a background field term. The Lagrangian is

$$L = \sum_{\rm sp} \int d\Lambda \, f \, L_p + \int d\mathcal{V} \, \frac{\rho_M c^2}{2B^2} \, |\nabla_\perp \phi|^2 \tag{B23}$$

where ρ_M is given by a static profile (or constant), and L_p is of the form given in Eq. (1), with

$$H = \frac{p_z^2}{2m} + \mu B + e\phi \tag{B24}$$

The corresponding polarisation equation is

$$\nabla \cdot \frac{\rho_M c^2}{B^2} \nabla_\perp \phi + \sum_{\rm sp} \int d\mathcal{W} \, ef = 0 \tag{B25}$$

This is the same model as in the ORB code mentioned above, except we simplify by neglecting $(1 - J_0)$ hence FLR effects.

The particle drifts are given by Eq. (4), with spatial part

$$B_\parallel^* \dot{\mathbf{R}} = \nabla\phi \cdot \frac{c\mathbf{F}}{B} + \mu \nabla B \cdot \frac{c}{e} \frac{\mathbf{F}}{B} - \frac{p_z^2}{m} \left(\nabla \cdot \frac{c}{e} \frac{\mathbf{F}}{B} \right) + \frac{p_z}{m} \mathbf{B} \tag{B26}$$

with the pieces identified as the ExB velocity, the grad-B and curvature drifts, and the parallel velocity. The radial component of this is then

$$\dot{V} = \nabla V \cdot \dot{\mathbf{R}} = (v_E)^V + (v_{\nabla B})^V + (v_C)^V \tag{B27}$$

which we use below (the parallel piece does not contribute).

The steps to the momentum equation are the derivations of Eqs. (41,50,63) which remain unchanged, and the polarisation vector

$$\mathbf{P} = -\frac{\rho_M c^2}{B^2} \nabla_\perp \phi \tag{B28}$$

with the only difference to the MHD model at this point being the static ρ_M. Specifically, we still have

$$\frac{\partial}{\partial t} \langle fe \rangle = -\left\langle \nabla \cdot \frac{\partial}{\partial t} \frac{\rho_M c^2}{B^2} \nabla_\perp \phi \right\rangle = -\frac{\partial}{\partial V} \left\langle \frac{\partial P^V}{\partial t} \right\rangle \tag{B29}$$

with the background ρ_M. The cancellation in Eq. (79) remains intact, and we have Eq. (80). We evaluate

$$\left\langle fe\frac{\partial\phi}{\partial\varphi}\right\rangle = -\frac{\partial}{\partial V}\left\langle \rho_M \frac{c^2}{B^2}\frac{\partial\phi}{\partial\varphi}\nabla V\cdot\nabla\phi\right\rangle \tag{B30}$$

to obtain

$$\frac{\partial}{\partial t}\left\langle \rho_M (v_E)_\varphi + f p_z b_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle \nabla V\cdot\left(f p_z b_\varphi \dot{\mathbf{R}} - \rho_M \frac{c^2}{B^2}\frac{\partial\phi}{\partial\varphi}\nabla\phi\right)\right\rangle = 0 \tag{B31}$$

as the local toroidal momentum conservation equation, *i.e.*, the toroidal momentum transport equation. The pieces are easily identified as the ExB and parallel components of the toroidal momentum, the ExB/parallel Reynolds stress, and the toroidal component of the pure ExB Reynolds stress, all familiar effects. In the drifts, $(v_E)^V$ gives the ExB/parallel Reynolds stress, while $(v_{\nabla B})^V$ and $(v_C)^V$ give the grad-B and curvature drifts which comprise the neoclassical transport (to evaluate these, of course, one needs treatment of the collisions; cf. the review by Hinton and Hazeltine [43]). Except for the corrections arising through J_0, this is the toroidal momentum equation satisfied by the model used in the conventional numerical models mentioned above.

The recent NEMORB code [61] is the electromagnetic version of ORB. It reestablishes $A_\parallel$ at first order in H, placing the second order term alongside the polarisation field term. The version neglecting FLR effects is given by

$$L = \sum_{\text{sp}}\int d\Lambda\, f\, L_p + \int d\mathcal{V}\left(\frac{\rho_M c^2}{2B^2}\left|\nabla_\perp\phi\right|^2 - \frac{n_0 e^2}{2Mc^2}A_\parallel^2 - \frac{1}{8\pi}\left|\nabla_\perp A_\parallel\right|^2\right) \tag{B32}$$

with reduced mass $M = m_e M_i/(m_e + M_i)$ and with L_p as in Eq. (1), and H given by

$$H = \frac{p_z^2}{2m} + \mu B + e\left(\phi - \frac{p_z}{mc}A_\parallel\right) \tag{B33}$$

The polarisation equation is unchanged, given by Eq. (B25), and the induction equation is

$$\left(\frac{\omega_p^2}{c^2} - \nabla_\perp^2\right)A_\parallel = \sum_{\text{sp}}\int d\mathcal{W}\,\frac{4\pi e}{mc}\,p_z\, f \qquad\qquad \omega_p^2 = \frac{4\pi n_0 e^2}{M} \tag{B34}$$

These are the simplifications of Eqs. (14,25) of Ref. [7], respectively, descending accordingly from the simplification of H (see the discussion at the beginning of Sec. VII C).

The corresponding toroidal momentum transport equation is

$$\frac{\partial}{\partial t}\left\langle \rho_M (v_E)_\varphi + f p_z b_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle \nabla V\cdot\left(f p_z b_\varphi \dot{\mathbf{R}} - \rho_M \frac{c^2}{B^2}\frac{\partial\phi}{\partial\varphi}\nabla\phi + \frac{1}{4\pi}\frac{\partial A_\parallel}{\partial\varphi}\nabla A_\parallel\right)\right\rangle = 0 \tag{B35}$$

which merely adds the Maxwell stress as the last term. Other terms from the field equations are put into the form $\partial S/\partial\varphi$ with scalar S and are annihilated by the flux surface average. The neglect of FLR effects in this demonstration is merely for clarity; we may restore J_0 and expand it in a series of Laplacians to recover several correction effects following the procedure for Laplacian field variable dependence given in Sec. VII B. Hence, we have demonstrated that momentum conservation and transport in ORB and NEMORB, as well as other related codes, is on a solid foundation.

Appendix C: Torque due to a Charge Source

In exceptional cases the particle sources are not in charge balance. The exemplary case is ion orbit loss [62, 63], in which the ions on large trapped (banana) orbits impact material surfaces while the electrons remain confined. Strictly speaking this is a transport effect and is accounted for by the drifts term and a loss flux through the boundary. In practice, however, it is modelled by a localised loss term in the ion continuity equation. The corresponding vorticity transport equation is

$$\frac{\partial}{\partial t}\langle\Omega\rangle - \frac{\partial}{\partial V}\left\langle fe\dot{V}\right\rangle = \langle S_\Omega\rangle \qquad\qquad S_\Omega = -e_i\left.\frac{df_i}{dt}\right|_{\rm loss} \tag{C1}$$

Since S_Ω is a scalar quantity we may specify

$$\nabla_\perp^2 s = S_\Omega \tag{C2}$$

with $s = \partial s/\partial V = 0$ on the magnetic axis ($V = 0$), without loss of generality. With $\nabla\cdot\mathbf{P} = -\Omega$ we have

$$\frac{\partial}{\partial V}\left\langle\frac{\partial P^V}{\partial t} + fe\dot{V} - \nabla V\cdot\nabla s\right\rangle = 0 \tag{C3}$$

so that the flux surface average quantity vanishes as before.

In the toroidal momentum continuity equation the terms which involve A_φ are

$$\begin{aligned}\frac{\partial}{\partial t}\frac{A_\varphi}{c}\langle fe\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle + \frac{A_\varphi}{c}\langle S_\Omega\rangle = \frac{\partial}{\partial t}\left\langle -\frac{1}{c}\mathbf{P}\cdot\nabla A_\varphi\right\rangle - \left\langle\frac{1}{c}\nabla s\cdot\nabla A_\varphi\right\rangle \\ - \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle\frac{\partial P^V}{\partial t} + fe\dot{V} + \nabla V\cdot\nabla s\right\rangle\end{aligned} \tag{C4}$$

The terms on the last line vanish. With the manipulations which find

$$\nabla s\cdot\nabla A_\varphi = R^2\nabla\varphi\times(\nabla s\times\nabla\varphi)\cdot\nabla A_\varphi = R^2\mathbf{B}\times\nabla s\cdot\nabla\varphi \tag{C5}$$

we find

$$\frac{\partial}{\partial t}\frac{A_\varphi}{c}\langle fe\rangle+\frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle+\frac{A_\varphi}{c}\langle S_\Omega\rangle=\frac{\partial}{\partial t}\left\langle-\frac{1}{c}R^2\mathbf{B}\times\mathbf{P}\cdot\nabla\varphi\right\rangle-\left\langle\frac{1}{c}R^2\mathbf{B}\times\nabla s\cdot\nabla\varphi\right\rangle \quad \text{(C6)}$$

In the case of the MHD Hamiltonian $\mathbf{P}=-\rho_M(c^2/B^2)\nabla_\perp\phi$ and the first term gives the ExB covariant toroidal momentum (equivalent to toroidal angular momentum). Hence the second term can be represented as a toroidal torque

$$\mathbf{T}=\frac{1}{c}\mathbf{B}\times\nabla s \qquad\qquad R^2\mathbf{T}\cdot\nabla\varphi=T_\varphi \quad \text{(C7)}$$

and the vorticity source is given by

$$S_\Omega=\nabla\cdot\frac{c}{B^2}\mathbf{T}\times\mathbf{B} \quad \text{(C8)}$$